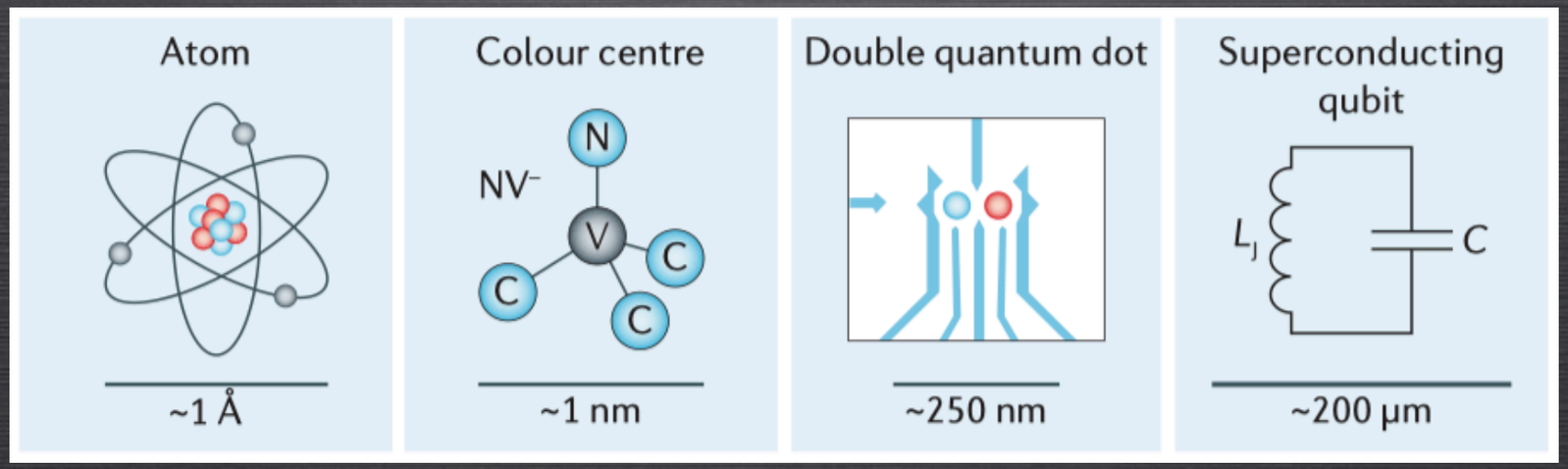
Superconducting quantum circuits

(circuit quantum electrodynamics, cQED)



Rok Žitko

F1, Institut "Jožef Stefan"

FMF, Univerza v Ljubljani

UL FMF, specialistični seminar iz teoretične fizike, Feb 2022

Figure source: Burkard et al. (2020)

- Hamiltonian formalism for electrical circuits
- Quantum description of electrical circuits
- Quantum harmonic LC oscillator
- Josephson junction and non-linear effects
- Superconducting quantum interference device (SQUID)
- Superconducting qubits, transmon
- Resonators and dispersive read-out
- Hybrid superconductor-semiconductor devices
- Outlook and perspectives

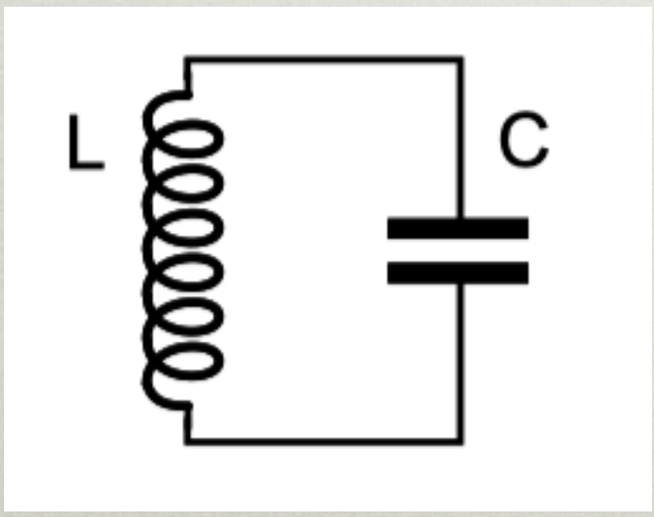
Quantum mechanics of macroscopic objects

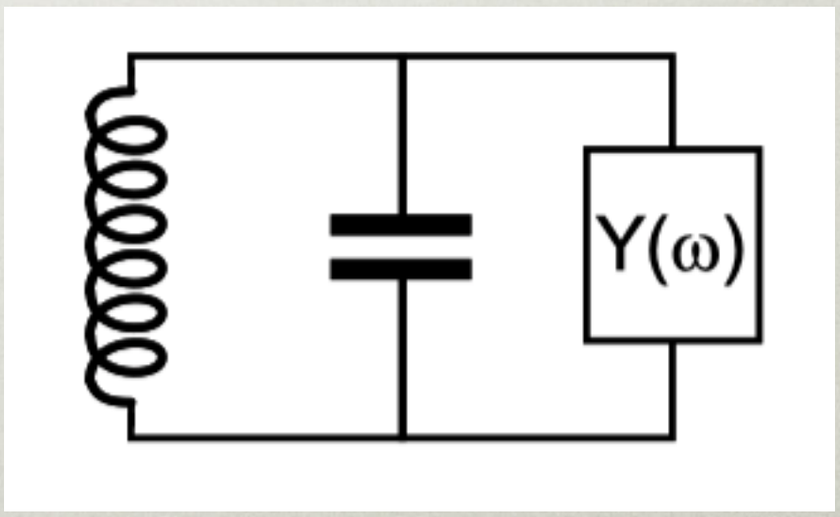
Mesoscopic systems: between microscopic quantum particles and macroscopic classical objects

Macroscopic: large number of atoms

Quantum: coherent collective degrees of freedom

Artificial: parameters can be tailored by design





$$L = 1\,\mathrm{nH} \quad C = 10\,\mathrm{pF} \quad \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \approx 1.6\,\mathrm{GHz}$$

Figure source: Vool, Devoret



Series on Directions in Condensed Matter Physics: Volume 1

Directions in Condensed Matter Physics

Memorial Volume in Honor of Shang-Keng Ma

https://doi.org/10.1142/0170 | August 1986

Pages: 268

Edited By: G Grinstein (*IBM*) and G Mazenko (*Chicago*)

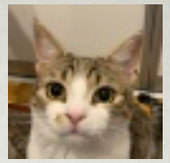
Tools Share Recommend to Library

This volume collects several in-depth articles giving lucid discussions on new developments in statistical and condensed matter physics. Many, though not all, contributors had been in touch with the late S-K Ma. Written by some of the world's experts and originators of new ideas in the field, this book is a must for all researchers in theoretical physics. Most of the articles should be accessible to diligent graduate students and experienced readers will gain from the wealth of materials contained herein.

Contents:

- Percolation *(A Aharony)*
- Complex Analytic Methods for Viscous Flows in Two Dimensions *(D Bensimon et al.)*
- Pattern Recognition by Flexible Coils *(P G de Gennes)*
- Physics of Mesoscopic Systems *(Y Imry)*
- Models of Pattern Formation in First-Order Phase Transitions *(J S Langer)*
- Quantum mechanics at the Macroscopic Level *(A J Leggett)*

$$\psi(x,t) = \alpha\,\psi_1(x,t) + \beta\,\psi_2(x,t)$$



Quantum superposition of distinct macroscopic states

Jonathan R. Friedman, Vijay Patel, W. Chen, S. K. Tolpygo & J. E. Lukens

Department of Physics and Astronomy, The State University of New York, Stony Brook, New York 11794-3800, USA

In 1935, Schrödinger[1] attempted to demonstrate the limitations of quantum mechanics using a thought experiment in which a cat is put in a quantum superposition of alive and dead states. The idea remained an academic curiosity until the 1980s when it was proposed[2–4] that, under suitable conditions, a macroscopic object with many microscopic degrees of freedom could behave quantum mechanically, provided that it was sufficiently decoupled from its environment. Although much progress has been made in demonstrating the macroscopic quantum behaviour of various systems such as superconductors[5–9], nanoscale magnets[10–12], laser-cooled trapped ions[13], photons in a microwave cavity[14] and C_{60} molecules[15], there has been no experimental demonstration of a quantum superposition of truly macroscopically distinct states. Here we present experimental evidence that a superconducting quantum interference device (SQUID) can be put into a superposition of two magnetic-flux states: one corresponding to a few microamperes of current flowing clockwise, the other corresponding to the same amount of current flowing anticlockwise.

Nature (2000)

Hamiltonian description of an electrical circuit

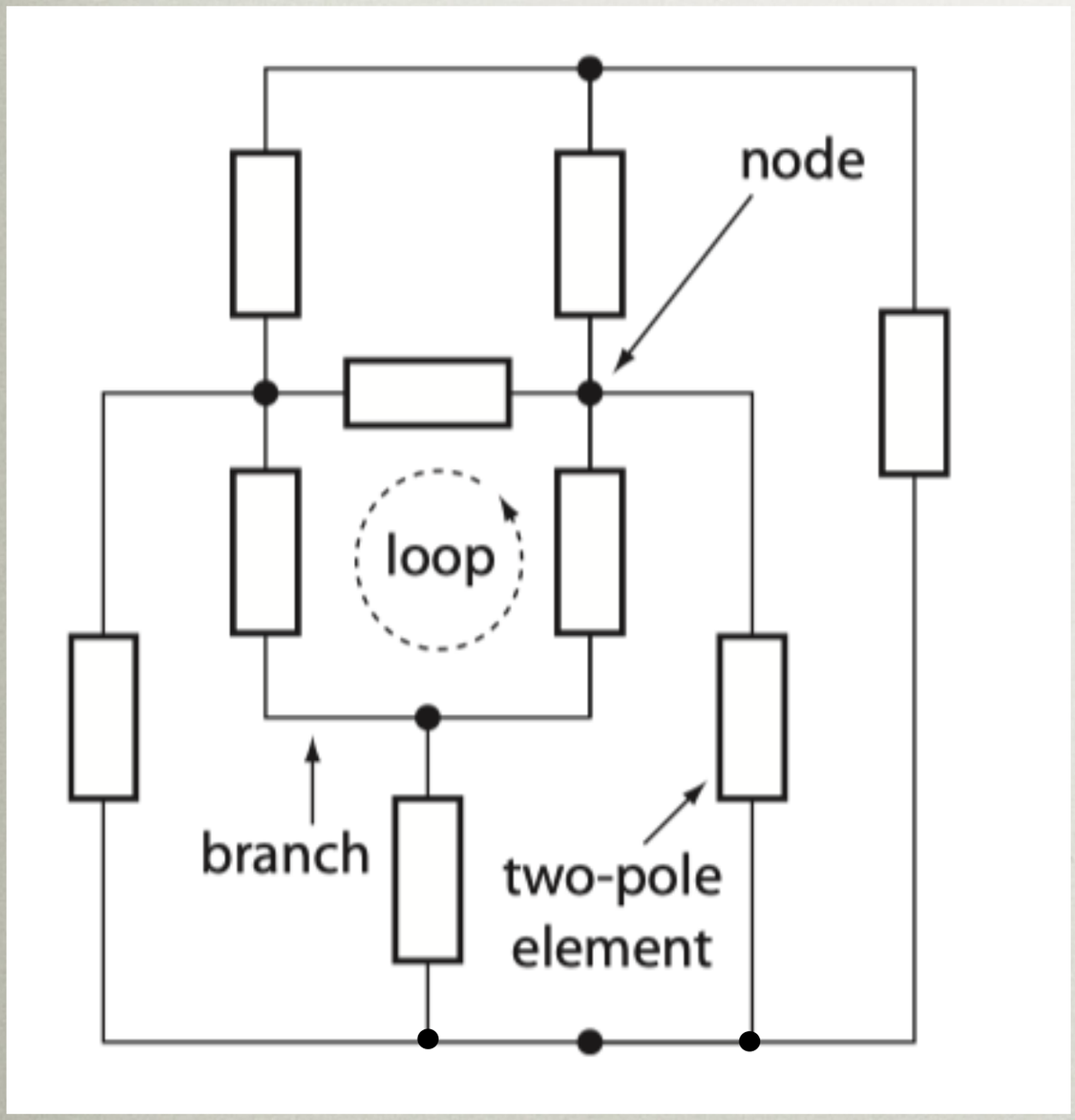


Figure source: Vool, Devoret

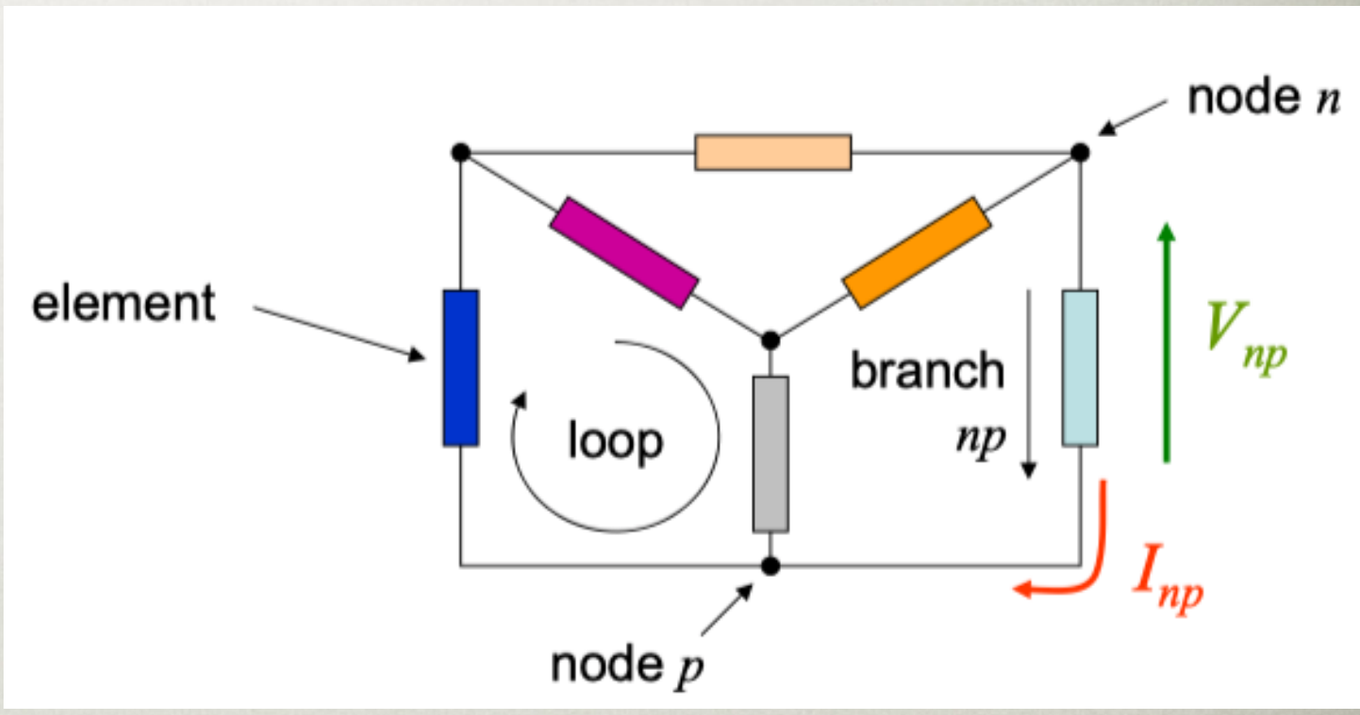


Figure source: Devoret lecture

LUMPED-ELEMENT MODEL

The lumped-element model of electronic circuits makes the simplifying assumption that the attributes of the circuit, resistance, capacitance, inductance, and gain, are concentrated into idealized electrical components; resistors, capacitors, and inductors, etc. joined by a network of perfectly conducting wires.

The lumped-element model is valid whenever $L_c \ll \lambda$, where L_c denotes the circuit's characteristic length, and λ denotes the circuit's operating wavelength. Otherwise, when the circuit length is on the order of a wavelength, we must consider more general models, such as the distributed-element model (including transmission lines), whose dynamic behaviour is described by Maxwell's equations. Another way of viewing the validity of the lumped-element model is to note that this model ignores the finite time it takes signals to propagate around a circuit. Whenever this propagation time is not significant to the application the lumped-element model can be used. This is the case when the propagation time is much less than the period of the signal involved. However, with increasing propagation time there will be an increasing error between the assumed and actual phase of the signal which in turn results in an error in the assumed amplitude of the signal. The exact point at which the lumped-element model can no longer be used depends to a certain extent on how accurately the signal needs to be known in a given application.

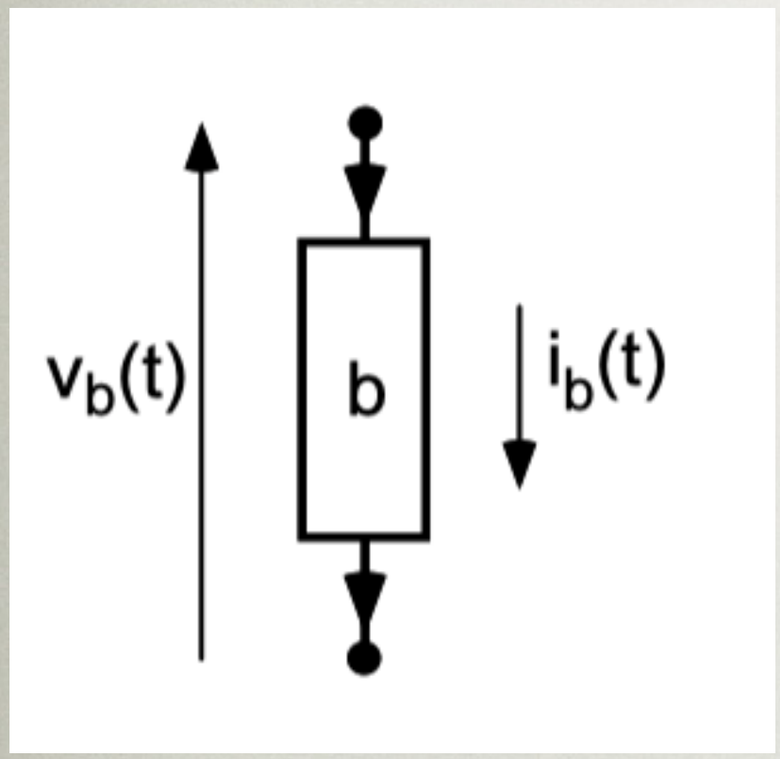
Real-world components exhibit non-ideal characteristics which are, in reality, distributed elements but are often represented to a first-order approximation by lumped elements. To account for leakage in capacitors for example, we can model the non-ideal capacitor as having a large lumped resistor connected in parallel even though the leakage is, in reality distributed throughout the dielectric. Similarly a wire-wound resistor has significant inductance as well as resistance distributed along its length but we can model this as a lumped inductor in series with the ideal resistor.

1 mm ~ 300 GHz

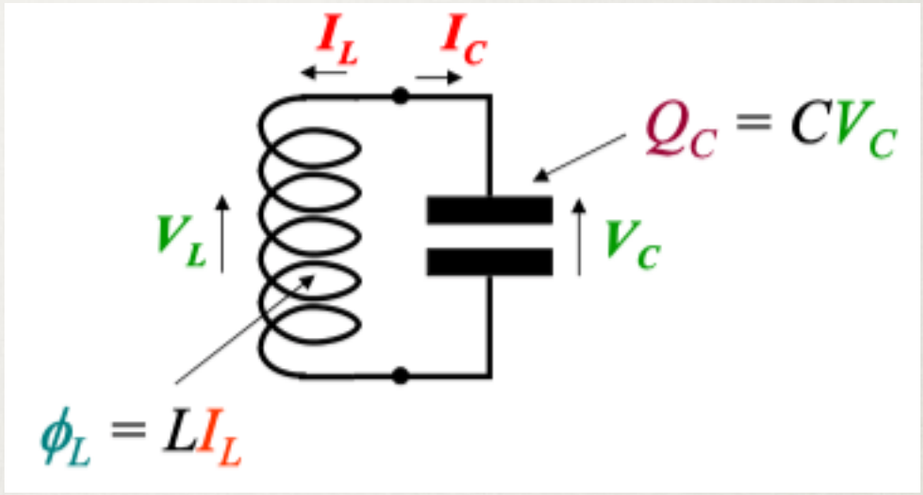
Source: Wikipedia

Dynamical variables

branch variables



example:



branch flux will turn out to be a position-like variable!

branch voltage

$$v_b(t) = \int_{\text{begining of } b}^{\text{end of } b} \mathbf{E}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{r}$$

$$\phi_b(t) = \int_{-\infty}^{t} v_b(t')\mathrm{d}t'$$

branch flux

branch current

$$i_b(t) = \frac{1}{\mu_0} \oint_{\text{around } b} \mathbf{B}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{r}$$

$$Q_b(t) = \int_{-\infty}^{t} i_b(t')\mathrm{d}t'$$

branch charge

Figure source: Vool, Devoret; Devoret lectures

inductance: $V = L\, dI/dt$

capacitance: $I = C\, dV/dt$

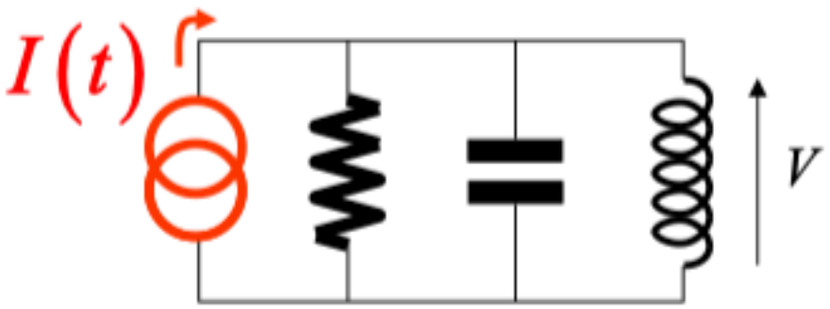
resistance: $V = R\, I$

$$\mathrm{d}\phi/\mathrm{d}t = L\mathrm{d}I/\mathrm{d}t \;\rightarrow\; \phi = LI$$

$$I = C\mathrm{d}^2\phi/\mathrm{d}t^2$$

$$\mathrm{d}\phi/\mathrm{d}t = RI$$

example : RCL circuit



$\phi(t)$: flux thru inductance

$\dot{\phi}(t)$: voltage V across inductance

current thru inductance ϕ / L

current thru resistance $\dot{\phi} / R$

current thru capacitance $C\ddot{\phi}$

$$C\ddot{\phi} + \frac{\dot{\phi}}{R} + \frac{\phi}{L} = I(t)$$

ETC...

Figure source: Devoret lecture

Energy absorbed by an element

$$\mathcal{E}_b(t) = \int_{-\infty}^{t} v_b(t') i_b(t') \mathrm{d}t'$$

Lagrangian $L = T - V$

velocity-like part / position-like part

Reactive elements: loss-less, all instantaneous power absorbed is eventually returned to the circuit.

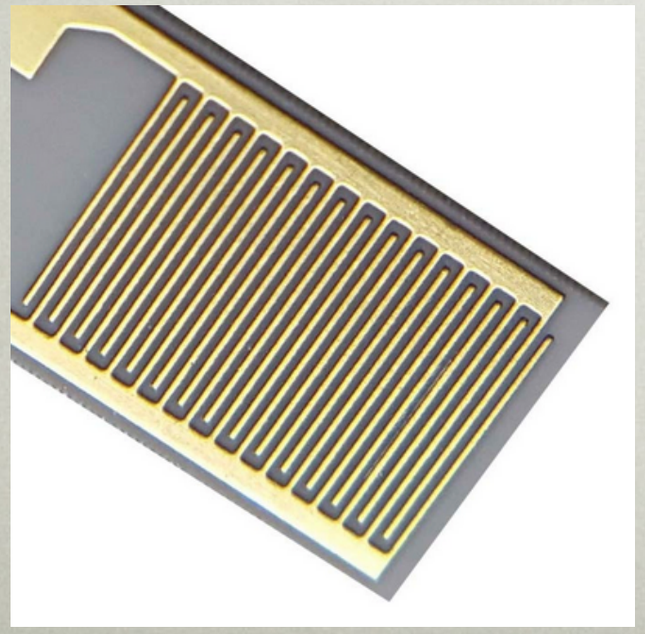
Figure source: Devoret lecture

Capacitive elements

$$v(t) = f[Q(t)]$$

capacitance $$C(Q) = \left[\frac{\mathrm{d}f}{\mathrm{d}Q}\right]^{-1}$$

linear capacitance $$v(t) = [Q(t) - Q_{\mathrm{offset}}]/C$$



interdigitated capacitor

$$\mathcal{E}(t) = \frac{1}{2C}[Q(t) - Q_{\mathrm{offset}}]^2$$

$$\mathcal{E}(t) = \frac{C}{2}\dot{\phi}^2$$ velocity-like part

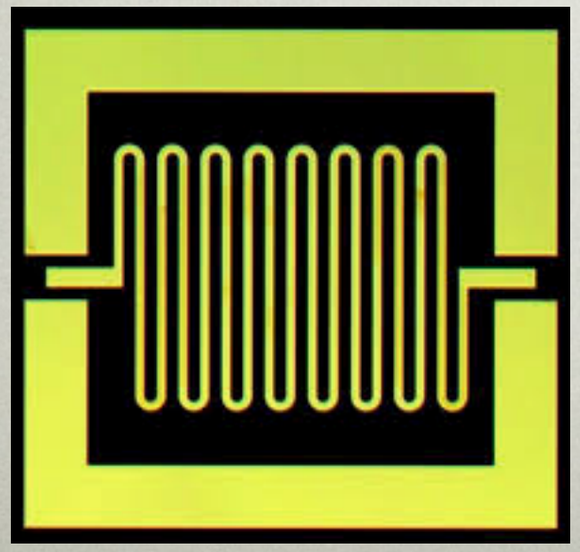
Inductive elements

$$i(t) = g[\Phi(t)]$$

inductance $$L(\Phi) = \left[\frac{\mathrm{d}g}{\mathrm{d}\Phi}\right]^{-1}$$

linear inductance $$i(t) = [\Phi(t) - \Phi_{\mathrm{offset}}]/L$$

$$\mathcal{E}(t) = \frac{1}{2L}[\Phi(t) - \Phi_{\mathrm{offset}}(t)]^2$$ position-like part



meandering inductor

Degrees of freedom

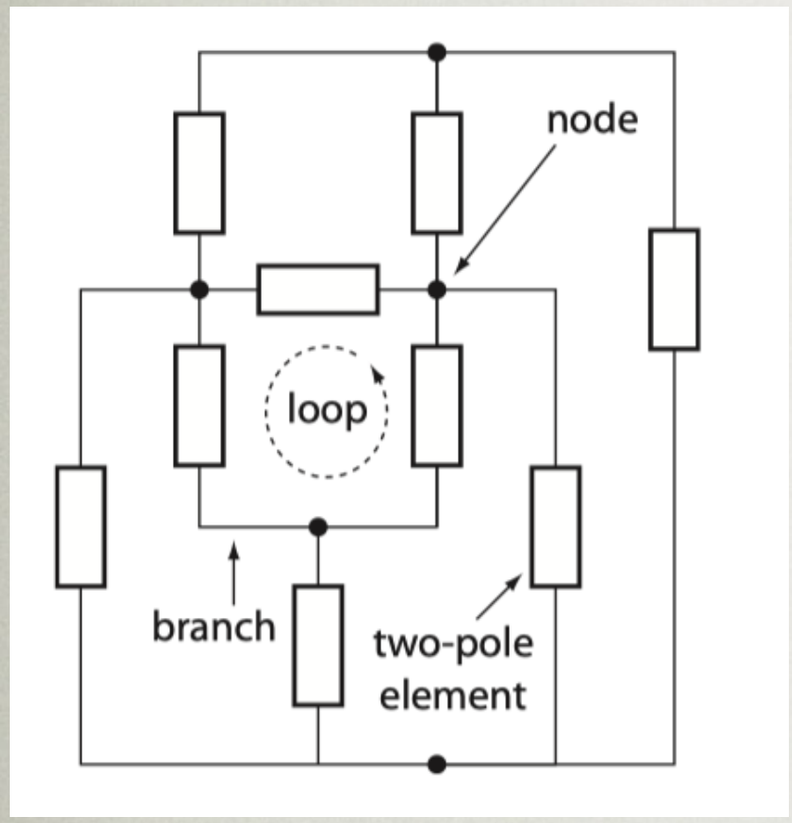


Figure source: Vool, Devoret

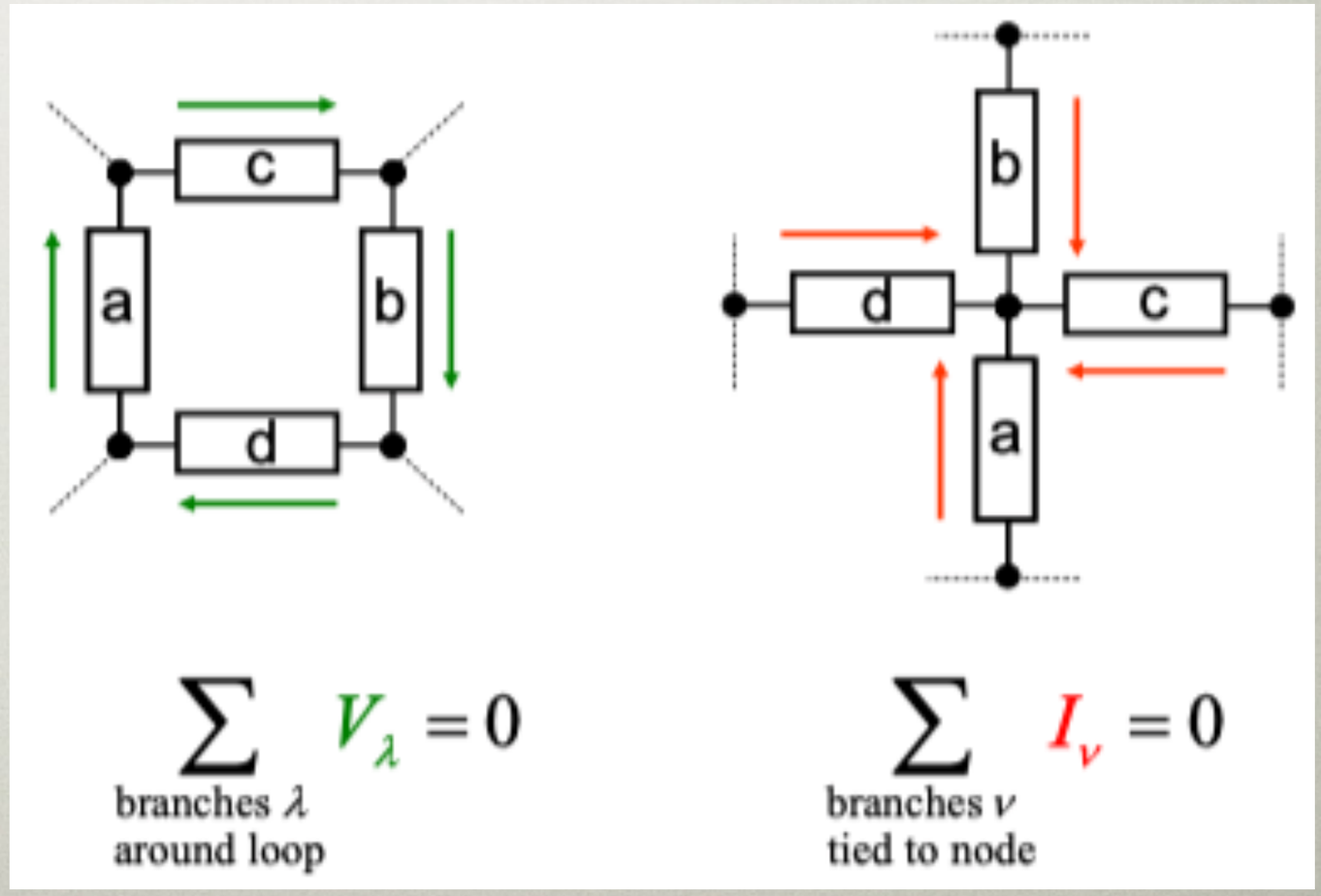
flux & charge formulation

Kirchhoff's laws

$$\sum_{\text{all } b \text{ around } l} \Phi_b = \tilde{\Phi}_l$$

$$\sum_{\text{all } b \text{ arriving at } n} Q_b = \tilde{Q}_n$$

voltage & current formulation (steady state)



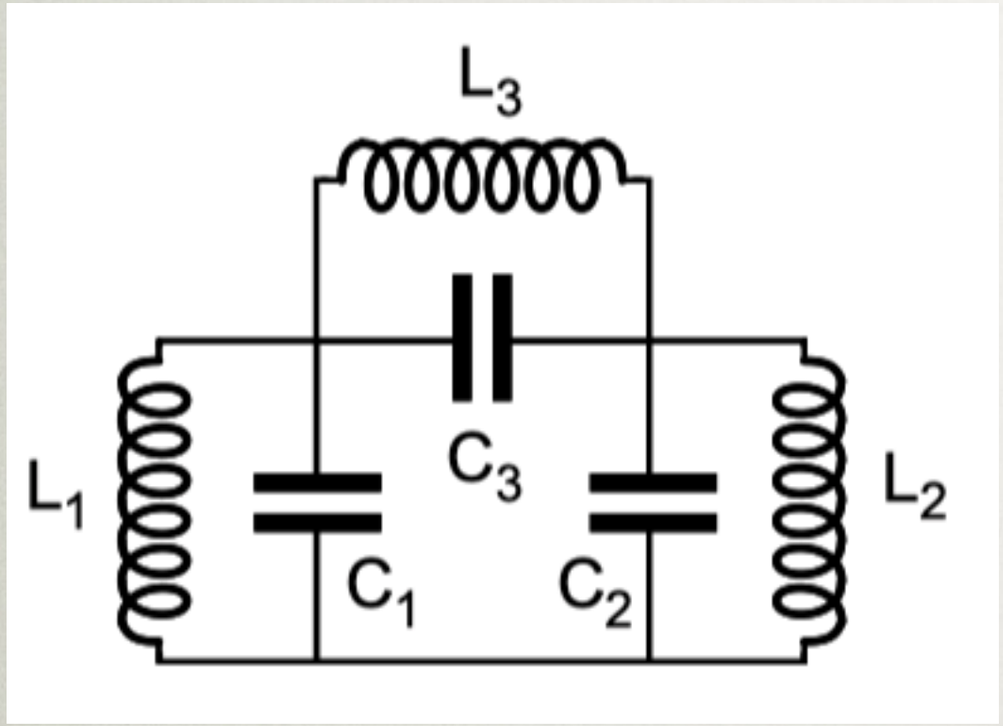
$$\sum_{\text{branches } \lambda \text{ around loop}} V_\lambda = 0$$

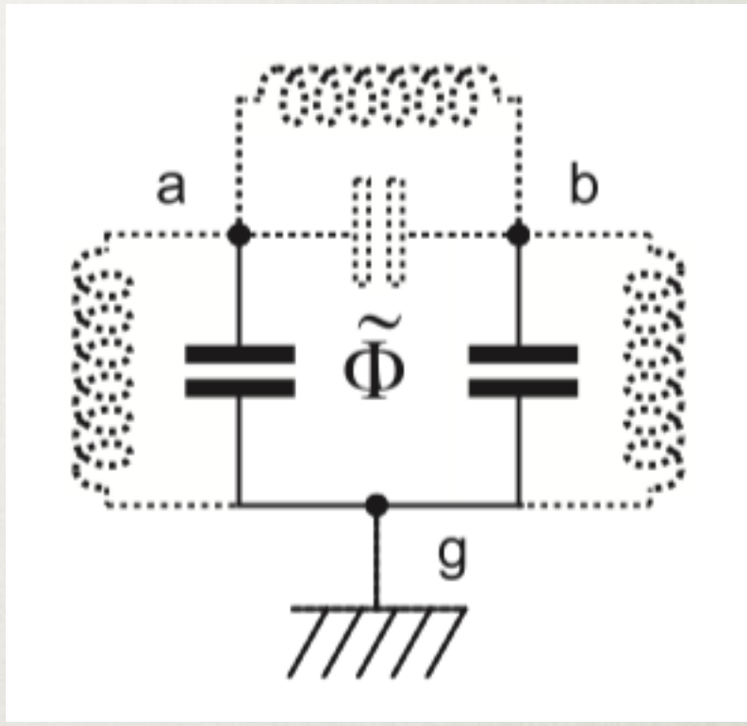
$$\sum_{\text{branches } \nu \text{ tied to node}} I_\nu = 0$$

Figure source: Devoret lecture

Method of nodes, spanning tree

Many possibilities for spanning tree; similar to fixing a gauge.





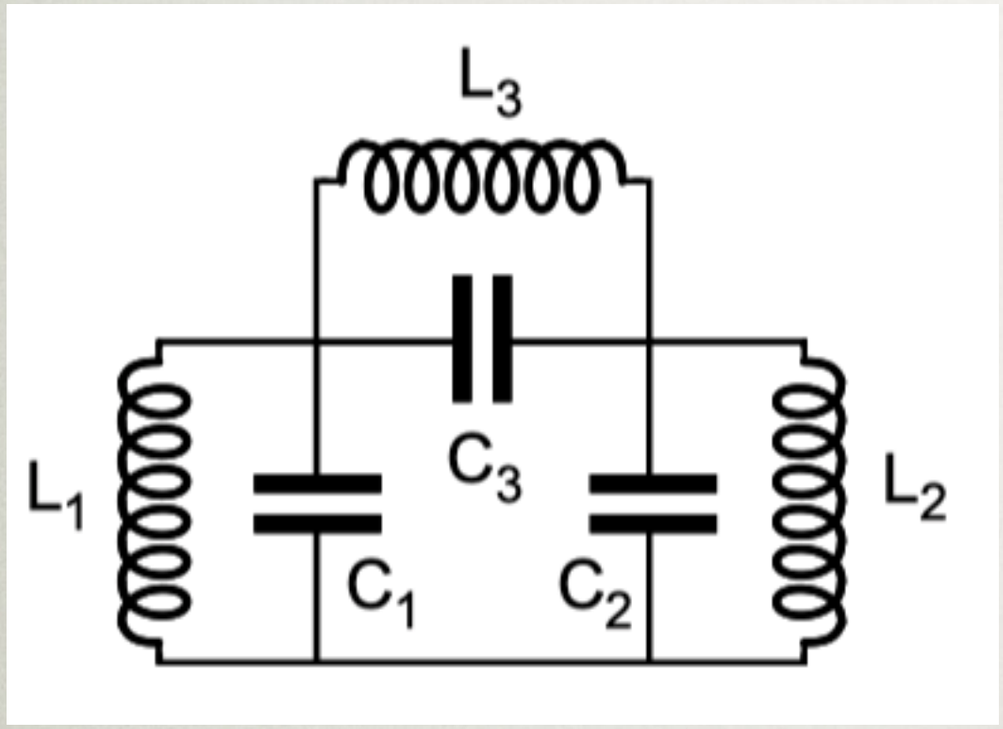
$$\Phi_{b\in T} = \phi_n - \phi_{n'}$$ branches in the tree

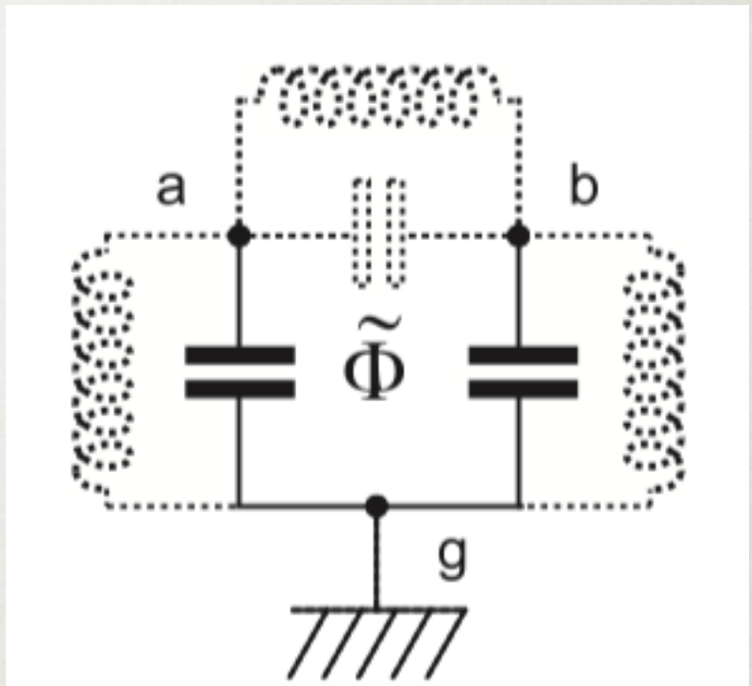
$$\Phi_{b\in \bar{T}} = \phi_n - \phi_{n'} + \tilde{\Phi}_b$$ branches not in the tree

Applicable to any circuit that includes capacitances, inductances (and Josephson junctions).

Always works, because spurious capacitances exist between all circuit nodes.

Figure source: Vool, Devoret





$$\mathcal{L}\left(\phi_a, \dot{\phi}_a, \phi_b, \dot{\phi}_b\right) = \frac{C_1\dot{\phi}_a^2}{2} + \frac{C_2\dot{\phi}_b^2}{2} + \frac{C_3\left(\dot{\phi}_a - \dot{\phi}_b\right)^2}{2} - \left[\frac{\phi_a^2}{2L_1} + \frac{\phi_b^2}{2L_2} + \frac{\left(\phi_a - \phi_b + \widetilde{\Phi}\right)^2}{2L_3}\right]$$

Figure source: Vool, Devoret

Lagrangian of a circuit

kinetic energy, associated with velocity-like variables

$$\mathcal{E}_{\text{kin}} = \frac{1}{2}\dot{\boldsymbol{\phi}}^T[\mathbf{C}]\dot{\boldsymbol{\phi}}$$

potential energy, associated with position-like variables

$$\mathcal{E}_{\text{pot}} = \frac{1}{2}\boldsymbol{\phi}^T[\mathbf{L}^{-1}]\boldsymbol{\phi} + \sum_b \frac{1}{L_b}(\phi_n - \phi_{n'})\tilde{\Phi}_b$$

Lagrangian $\mathcal{L} = \mathcal{E}_{\text{kin}} - \mathcal{E}_{\text{pot}}$

Euler-Lagrange EOMs $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}_n} = \frac{\partial \mathcal{L}}{\partial \phi_n}$

"dynamical matrix" $[\Omega^2] = [\mathbf{C}^{-1}][\mathbf{L}^{-1}]$

System of coupled oscillators.

$$-[\mathbf{C}]\ddot{\boldsymbol{\phi}} = [\mathbf{L}^{-1}]\boldsymbol{\phi}$$

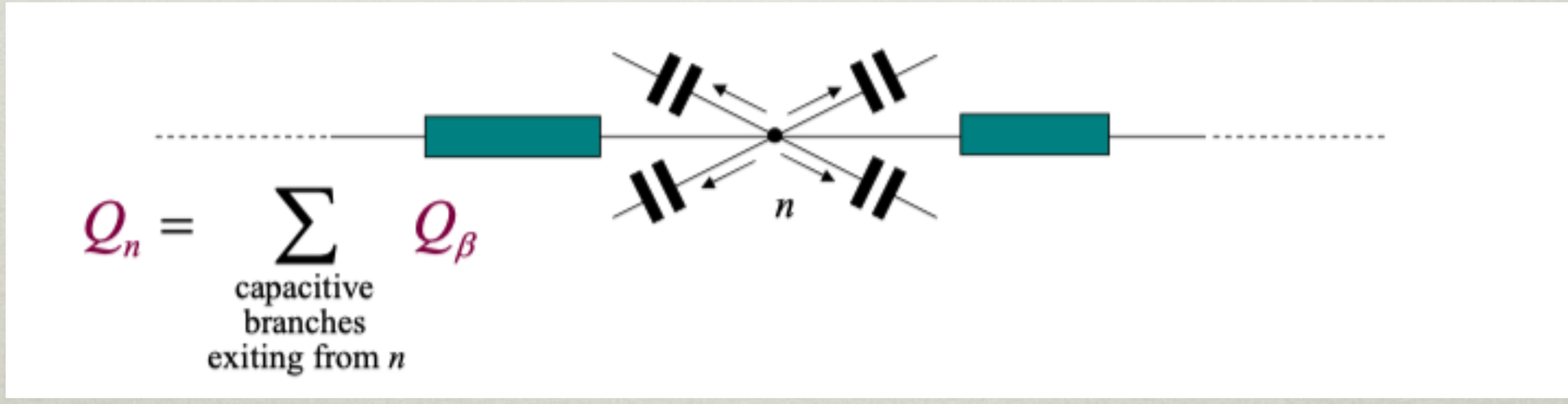
$$-\ddot{\boldsymbol{\phi}} = [\mathbf{C}^{-1}][\mathbf{L}^{-1}]\boldsymbol{\phi}$$

Conjugate variables

conjugate momenta = node charges!

$$q_n = \frac{\partial \mathcal{L}}{\partial \dot{\phi}_n}$$

$$\mathbf{q} = [\mathbf{C}]\dot{\boldsymbol{\phi}}$$



charge on the capacitor plates connected to a node

Figure source: Devoret lectures

Hamiltonian of a circuit

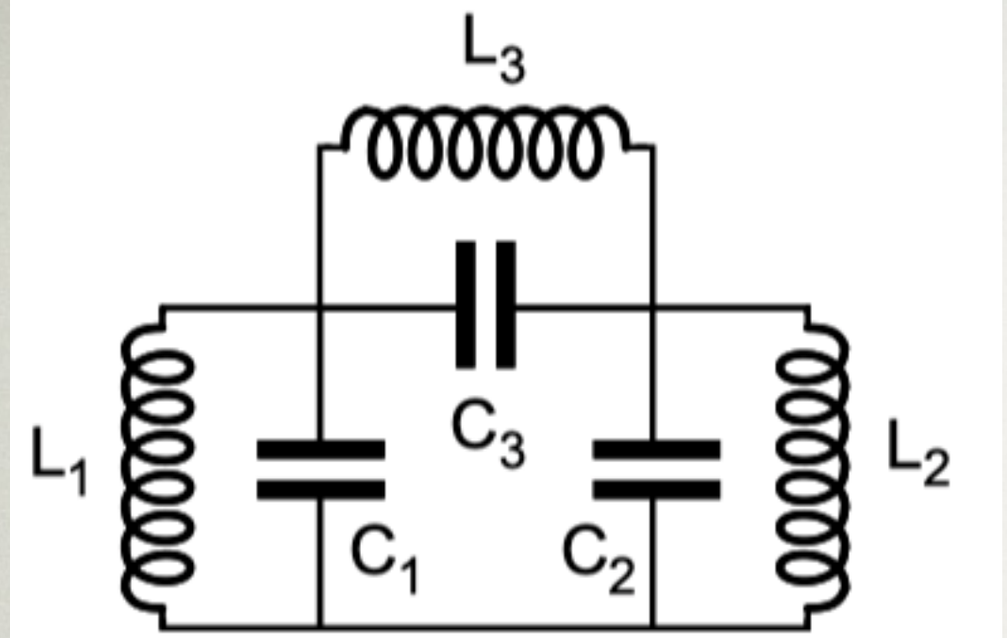
$$\mathcal{H} = \frac{1}{2}\mathbf{q}^T[\mathbf{C}^{-1}]\mathbf{q} + \mathcal{E}_{\text{pot}} \qquad \mathcal{E}_{\text{pot}} = \frac{1}{2}\boldsymbol{\phi}^T[\mathbf{L}^{-1}]\boldsymbol{\phi} + \sum_b \frac{1}{L_b}(\phi_n - \phi_{n'})\tilde{\Phi}_b$$

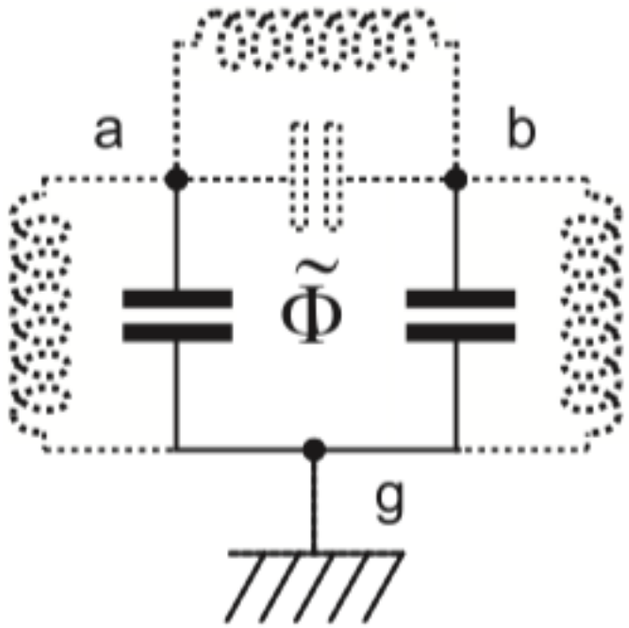
Hamiltons EOM's $\qquad \dot{\phi}_n = \dfrac{\partial \mathcal{H}}{\partial q_n} \qquad \dot{q}_n = -\dfrac{\partial \mathcal{H}}{\partial \phi_n}$

Poisson bracket of branch variables:

$$\{\Phi_b, Q_b\} = \sum_n \frac{\partial \Phi_b}{\partial \phi_n}\frac{\partial Q_b}{\partial q_n} - \frac{\partial Q_b}{\partial \phi_n}\frac{\partial \Phi_b}{\partial q_n} = \pm 1$$

+1 for capacitive branch
-1 for inductive branch





$$\mathcal{H}\left(\phi_a, q_a, \phi_b, q_b\right) = \frac{1}{C_1C_2 + C_1C_3 + C_2C_3}\left[\frac{\left(C_2 + C_3\right)q_a^2}{2} + \frac{\left(C_1 + C_3\right)q_b^2}{2} + C_3 q_a q_b\right] + \left[\frac{\phi_a^2}{2L_1} + \frac{\phi_b^2}{2L_2} + \frac{\left(\phi_a - \phi_b + \tilde{\Phi}\right)^2}{2L_3}\right]$$

```
In[4]:= {{C1 + C3, -C3}, {-C3, C2 + C3}}
        Inverse[%]

Out[4]= {{C1 + C3, -C3}, {-C3, C2 + C3}}

Out[5]= {{(C2 + C3)/(C1 C2 + C1 C3 + C2 C3), C3/(C1 C2 + C1 C3 + C2 C3)},
         {C3/(C1 C2 + C1 C3 + C2 C3), (C1 + C3)/(C1 C2 + C1 C3 + C2 C3)}}
```

Figure source: Vool, Devoret

IMPEDANCE, ADMITTANCE

complex amplitudes

$$Z = \frac{v}{i}$$

impedance

$$Z = R + jX$$

resistance | reactance

resistive $Z_R = R$

reactive $Z_L = j\omega L$

reactive $Z_C = \frac{1}{jC\omega}$

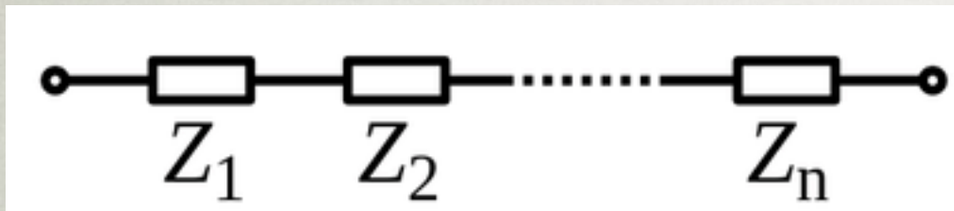
$$Y(\omega) \equiv \frac{1}{Z(\omega)}$$

admittance

$$Y = G + iB$$

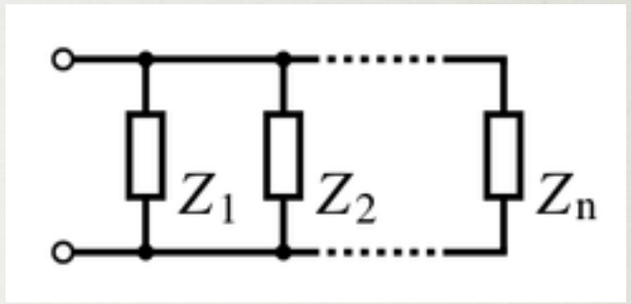
conductance | susceptance

ADDITION RULES



$$Z_{\text{eq}} = Z_1 + Z_2 + \cdots + Z_n$$

impedances add up in series



$$\frac{1}{Z_{\text{eq}}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_n}$$

$$Y_{\text{eq}} = Y_1 + Y_2 + \ldots + Y_n$$

admittances add up in parallel

Figure source: wikipedia

DIGRESSION: GENERALIZED IMPEDANCE & ADMITTANCE

$$Z(s) = \frac{\mathcal{L}\{v(t)\}}{\mathcal{L}\{i(t)\}} = \frac{V(s)}{I(s)}$$

$$s = \sigma + j\omega$$

Laplace transform

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t$$ mathematics/electronics convention

$$\mathcal{L}\{f\}(z) = \int_0^\infty f(t)e^{izt}\mathrm{d}t$$ physics convention

$$z = i\omega + \delta$$ complex frequency, $\delta > 0$

Automatically ensures correct causality properties!

kernel

$$i(t) = \int_{-\infty}^{+\infty} \mathrm{d}t'\, \tilde{Y}(t') v(t - t')$$

causality

$$\tilde{Y}(t') \equiv 0 \text{ for } t' < 0$$

$$Y(\omega + i\eta) = \int_{-\infty}^{+\infty} \tilde{Y}(t) \exp[i(\omega + i\eta)t]\, \mathrm{d}t$$

this is actually a Laplace transform!

$$Y(\omega) = \lim_{\eta \to 0, \eta > 0} Y(\omega + i\eta)$$

admittance — generalized admittance

Titchmarsh's theorem

Titchmarsh's theorem (named for E. C. Titchmarsh who included it in his 1937 work) makes precise the relationship between the boundary values of holomorphic functions in the upper half-plane and the Hilbert transform.[33] It gives necessary and sufficient conditions for a complex-valued square-integrable function $F(x)$ on the real line to be the boundary value of a function in the Hardy space $\mathrm{H}^2(U)$ of holomorphic functions in the upper half-plane U.

The theorem states that the following conditions for a complex-valued square-integrable function $F : \mathbb{R} \to \mathbb{C}$ are equivalent:

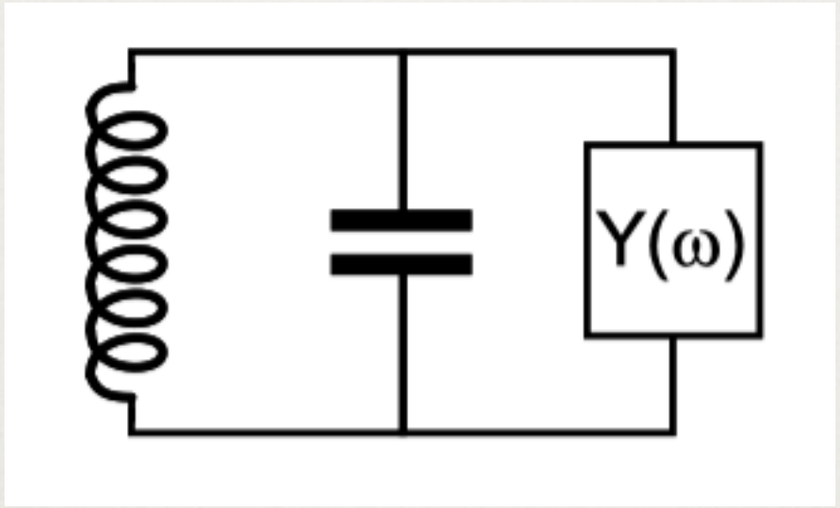
- $F(x)$ is the limit as $z \to x$ of a holomorphic function $F(z)$ in the upper half-plane such that

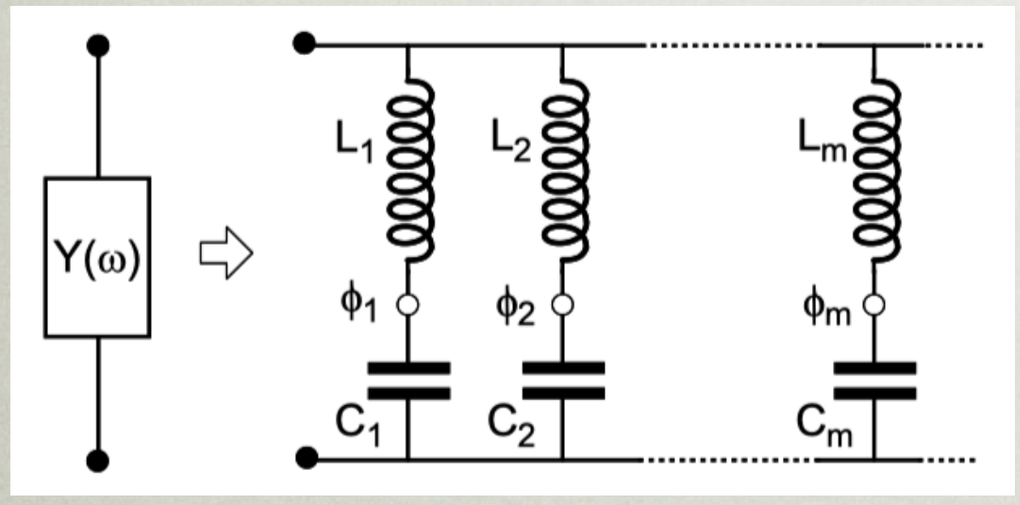
$$\int_{-\infty}^{\infty} |F(x + i\,y)|^2 \, \mathrm{d}x < K$$

- The real and imaginary parts of $F(x)$ are Hilbert transforms of each other.
- The Fourier transform $\mathcal{F}(F)(x)$ vanishes for $x < 0$.

cf. retarded Green's functions

Modelling dissipative elements





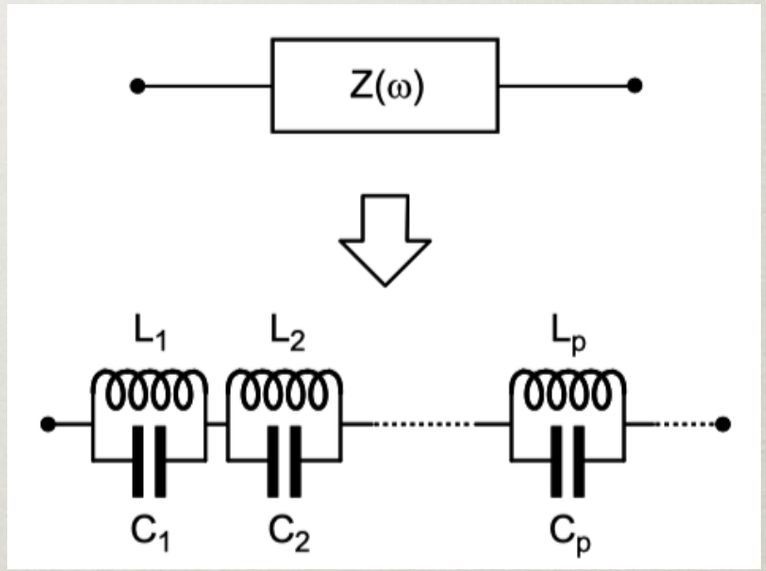


Figure source: Vool, Devoret

Admittance of LC element

$$Y_m(\omega) = \left[jL_m\omega + \frac{1}{jC_m\omega} \right]^{-1} \qquad \omega_m = \frac{1}{\sqrt{LC}} \qquad y_m = \sqrt{\frac{L}{C}}$$

$$Y_m(\omega) = y_m \left\{ \frac{\pi}{2}\omega_m[\delta(\omega-\omega_m)+\delta(\omega+\omega_m)] + \frac{i}{2}\left[\text{p.p.}\left(\frac{\omega_m}{\omega-\omega_m}\right) + \text{p.p.}\left(\frac{\omega_m}{\omega+\omega_m}\right)\right]\right\}$$

generalized admittance

similar to bosonic Green's functions!

$$Y[\omega+i\eta] = \frac{i}{L_0(\omega+i\eta)} + \lim_{\Delta\omega\to 0}\sum_{m=1}^{\infty}\left[jL_m(\omega+i\eta) + \frac{1}{jC_m(\omega+i\eta)}\right]^{-1}$$

$$\mathcal{H}_Y = \sum_m \left[\frac{q_m^2}{2C_m} + \frac{(\phi_m-\phi)^2}{2L_m}\right]$$

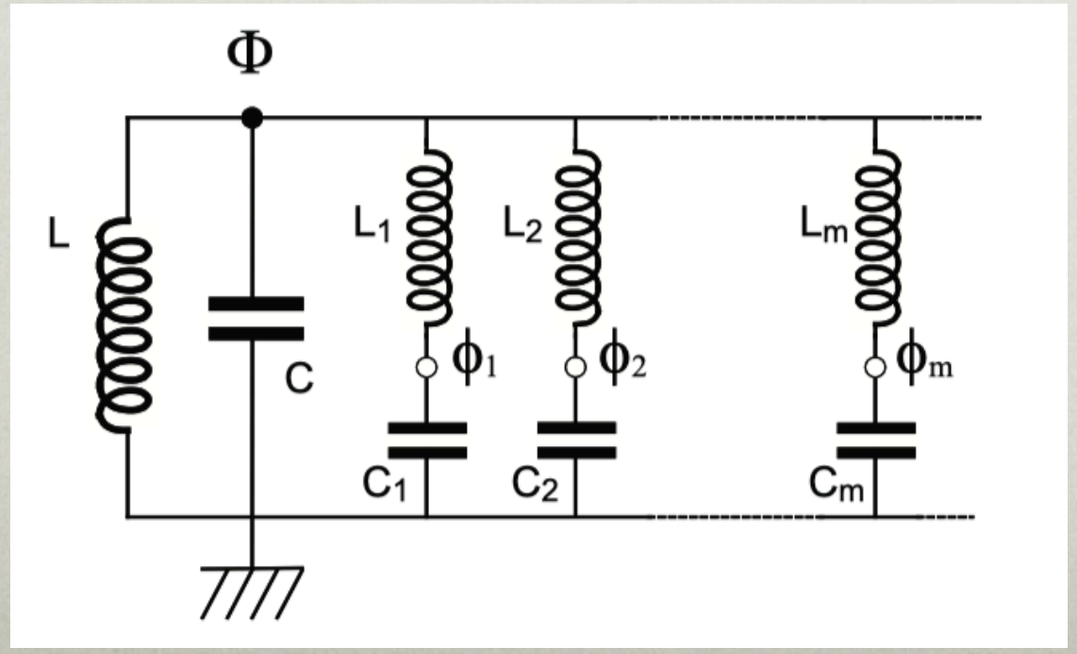


Figure source: Vool, Devoret

an infinite number of oscillator
can mimic a dissipative element!

cf. Caldeira-Leggett model

Quantum description of electrical circuits

Canonical quantization:

from variables to operators

$$\phi \to \hat{\phi}$$

$$q \to \hat{q}$$

$$\mathcal{H} \to \hat{\mathcal{H}}$$

Girvin: "microwaves, despite their name, really are particles"

Commutators [charge, flux]

$$[\hat{\phi}_n, \hat{q}_n] = i\hbar$$

node variables
(canonical coordinates)

$$\{A, B\} \to \frac{1}{i\hbar}[\hat{A}, \hat{B}]$$

$$[\hat{\Phi}_b, \hat{Q}_b] = \pm i\hbar$$

branch variables
(are <u>not</u> canonical coordinates)

CircuitQ: An open-source toolbox for superconducting circuits

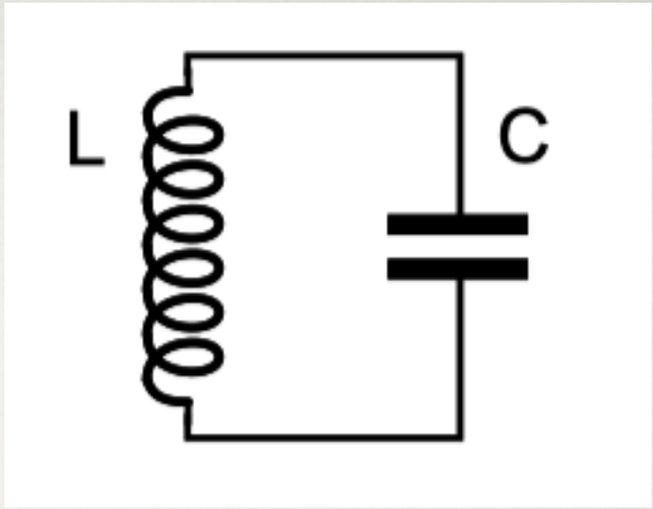
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(Dated: June 11, 2021)

We introduce CircuitQ, an open-source toolbox for the analysis of superconducting circuits implemented in Python. It features the automated construction of a symbolic Hamiltonian of the input circuit, as well as a dynamic numerical representation of this Hamiltonian with a variable basis choice. Additional features include the estimation of the T_1 lifetimes of the circuit states under various noise mechanisms. We review previously established circuit quantization methods and formulate them in a way that facilitates the software implementation. The toolbox is then showcased by applying it to practically relevant qubit circuits and comparing it to specialized circuit solvers. Our circuit quantization is both applicable to circuit inputs from a large design space and the software is open-sourced. We thereby add an important toolbox for the design of new quantum circuits for quantum information processing applications.

Quantum LC oscillator



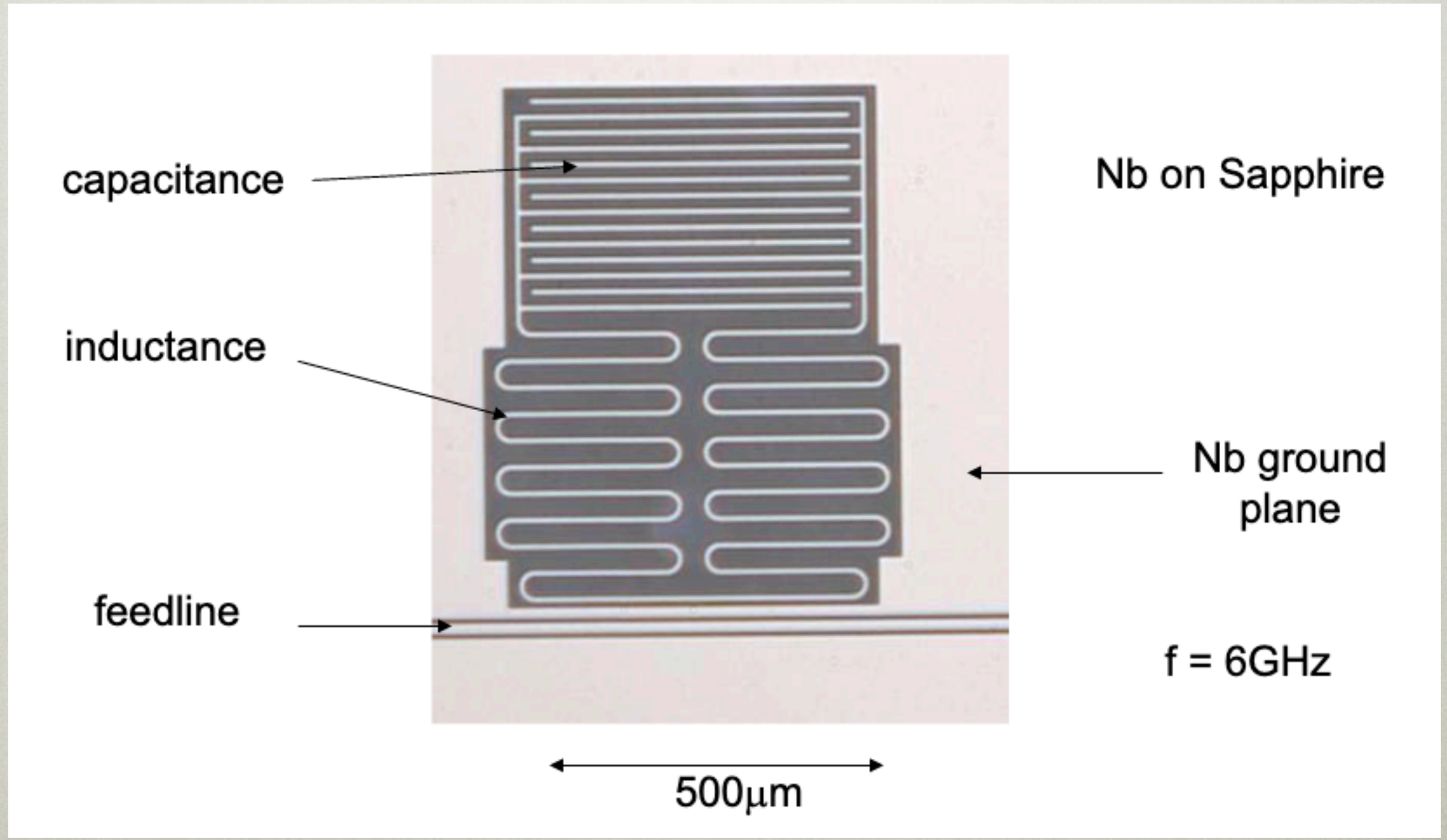
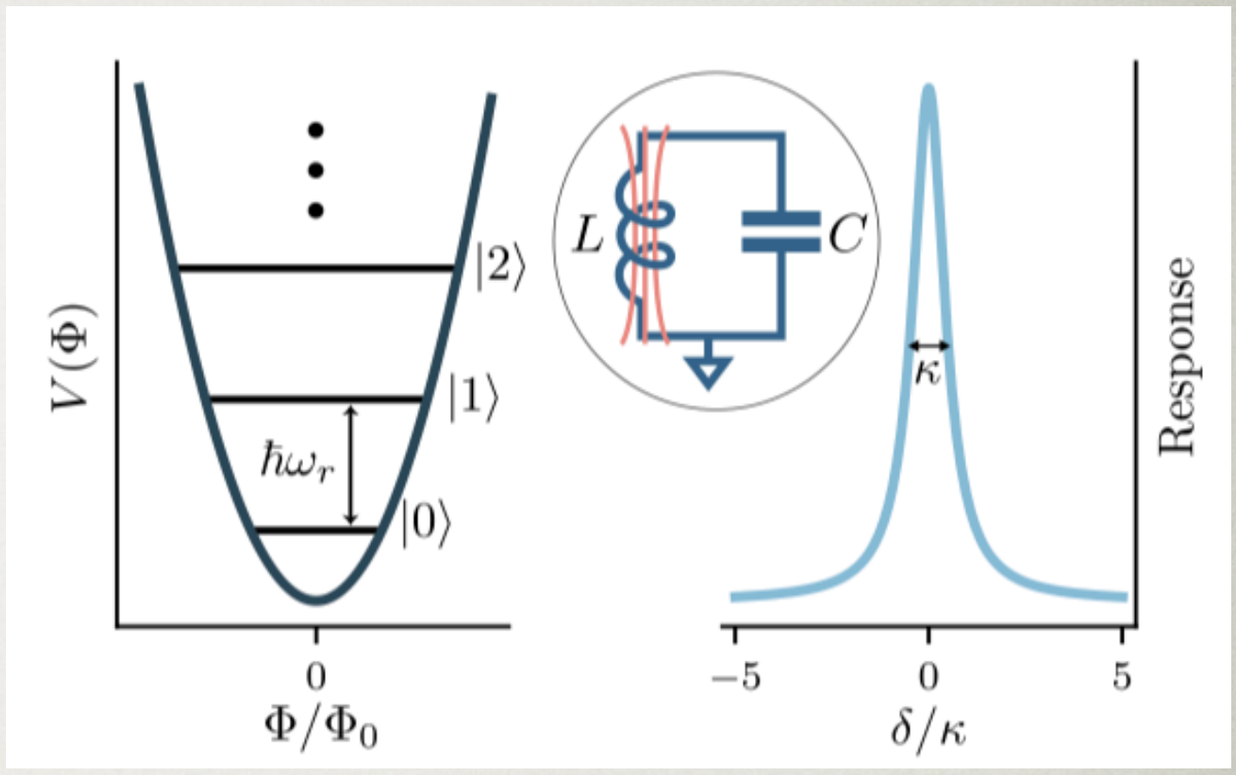


Figure source: Devoret lectures

Quantum LC oscillator

$$H = \frac{q^2}{2C} + \frac{\phi^2}{2L}$$

$$[c, c^\dagger] = 1$$



$$\phi = \phi_{\mathrm{ZPF}}(c + c^\dagger) \qquad \text{position-like}$$

$$q = \frac{1}{i} q_{\mathrm{ZPF}}(c - c^\dagger) \qquad \text{velocity-like}$$

$$c = \frac{\phi}{\phi_{\mathrm{ZPF}}} + i\frac{q}{q_{\mathrm{ZPF}}}$$

$$\mathcal{H} = \frac{\hbar\omega_0}{2}\left(c^\dagger c + cc^\dagger\right) = \hbar\omega_0\left(c^\dagger c + \frac{1}{2}\right)$$

$$\omega_0 = \sqrt{\frac{1}{LC}} \qquad \phi_{\mathrm{ZPF}} = \sqrt{\frac{\hbar Z_0}{2}}$$

$$Z_0 = \sqrt{\frac{L}{C}} \qquad q_{\mathrm{ZPF}} = \sqrt{\frac{\hbar}{2Z_0}}$$

$$\langle 0|\phi^2|0\rangle = \phi_{\mathrm{ZPF}}^2$$

$$\langle 0|q^2|0\rangle = q_{\mathrm{ZPF}}^2$$

Heisenberg's uncertainty principle

$$q_{\mathrm{ZPF}}\,\phi_{\mathrm{ZPF}} = \frac{\hbar}{2}$$

Figure source: Blais et al. RMP

$$\phi_{\mathrm{ZPF}} = \sqrt{\frac{\hbar Z_0}{2}} \qquad \phi_{\mathrm{ZPF}} = \sqrt{2\hbar\omega_0 L}$$

$$q_{\mathrm{ZPF}} = \sqrt{\frac{\hbar}{2Z_0}} \qquad q_{\mathrm{ZPF}} = \sqrt{2\hbar\omega_0 C}$$

characteristic impedance $\quad Z_0 = \sqrt{\frac{L}{C}}$

dimensionless characteristic impedance $\quad z = \frac{Z_0}{R_Q}$

superconducting resistance quantum $\quad R_Q \equiv \frac{h}{(2e)^2} \approx 6453.20\,\Omega$

$$Q_{\mathrm{ZPF}} = (2e)\sqrt{\frac{1}{4\pi z}}$$

$$\phi_{\mathrm{ZPF}} = \Phi_0\sqrt{\frac{z}{4\pi}}$$

(superconducting) magnetic flux quantum

$$\Phi_0 = \frac{h}{2e}$$

operation close to ground state

$$\hbar\omega_0 \gg k_B T$$

$5\,\mathrm{GHz}$ $10\,\mathrm{mK}$

$\sim 250\,\mathrm{mK}$

Approximate Conversion	Exact value	Formula	Note
1 uA = 300 pH	318 pH	L_J = \Phi_0 / 2 pi I_C	Josephoson inductance
1 GHz = 4 uV = 47 mK	4.11 uV	E = h \nu	Frequency to energy conversion for EM radiation
10 fF = 8 uV = 2 GHz		E = e^2 / 2 C	Charging energy
10 nA = 20 uV = 5 GHz	2 uV	E_J = h/2e * Ic/2pi	Josephson Energy (careful about factors of two?...)
1 K = 150 ueV		\Delta = 1.76 kTc	Superconducting gap as a function of Tc (Al = 1.2K)
10 kOhm = 230 nA		I_c = pi * Delta / 2 Rn	Ic Rn product for a junction with Tc = 10K
10 kOhm = 23 nA		I_c = pi * Delta / 2 Rn	Ic Rn product for a junction with Tc = 1K
100 uV = 1 K	1.16 K (86 uV)	E = e / k_B	
1 cm = 1 pF	1.11 pF	C = 4 \pi \epsilon d	Self capacitance of a sphere to infinity
100 aF = 0.80 mV		E = e^2 / 2 C	Charging energy
1 ns = 30 cm	30 cm	d = c / t	Speed of light
1 pF = 160 nV, 16 mV = 10 aF		V = q / C	Voltage on capacitor for a single electron
1 cm = 1 nH			Self inductance of wire per unit length?
1V @ 100 nm = 0.5 x 10^11 e/cm^2	0.55 x 10^11	Q/A = \epsilon V / d	Parallel plate capacitor, \epsilon = 1

https://nsweb.tn.tudelft.nl/~gsteele/conv/

Dilution fridge

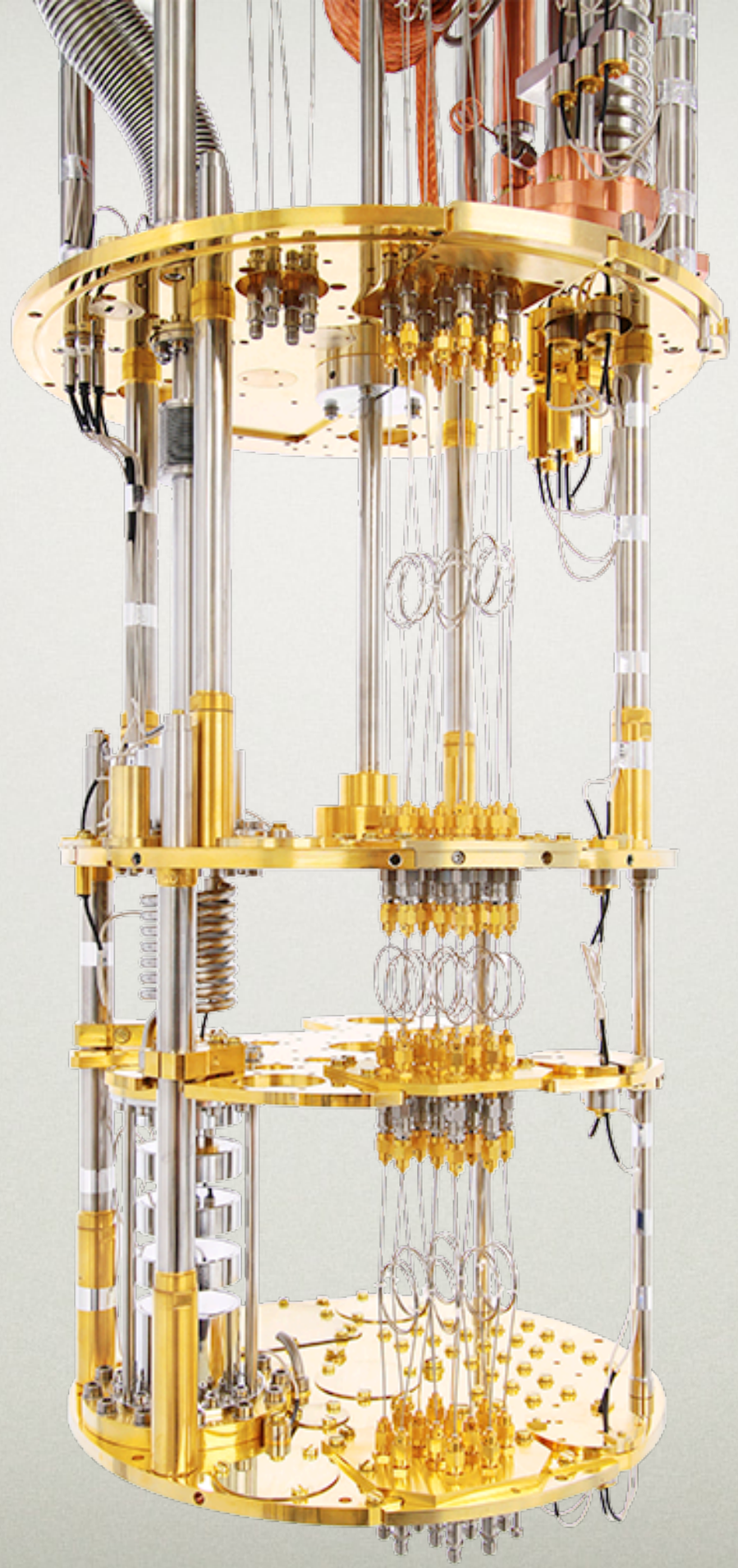
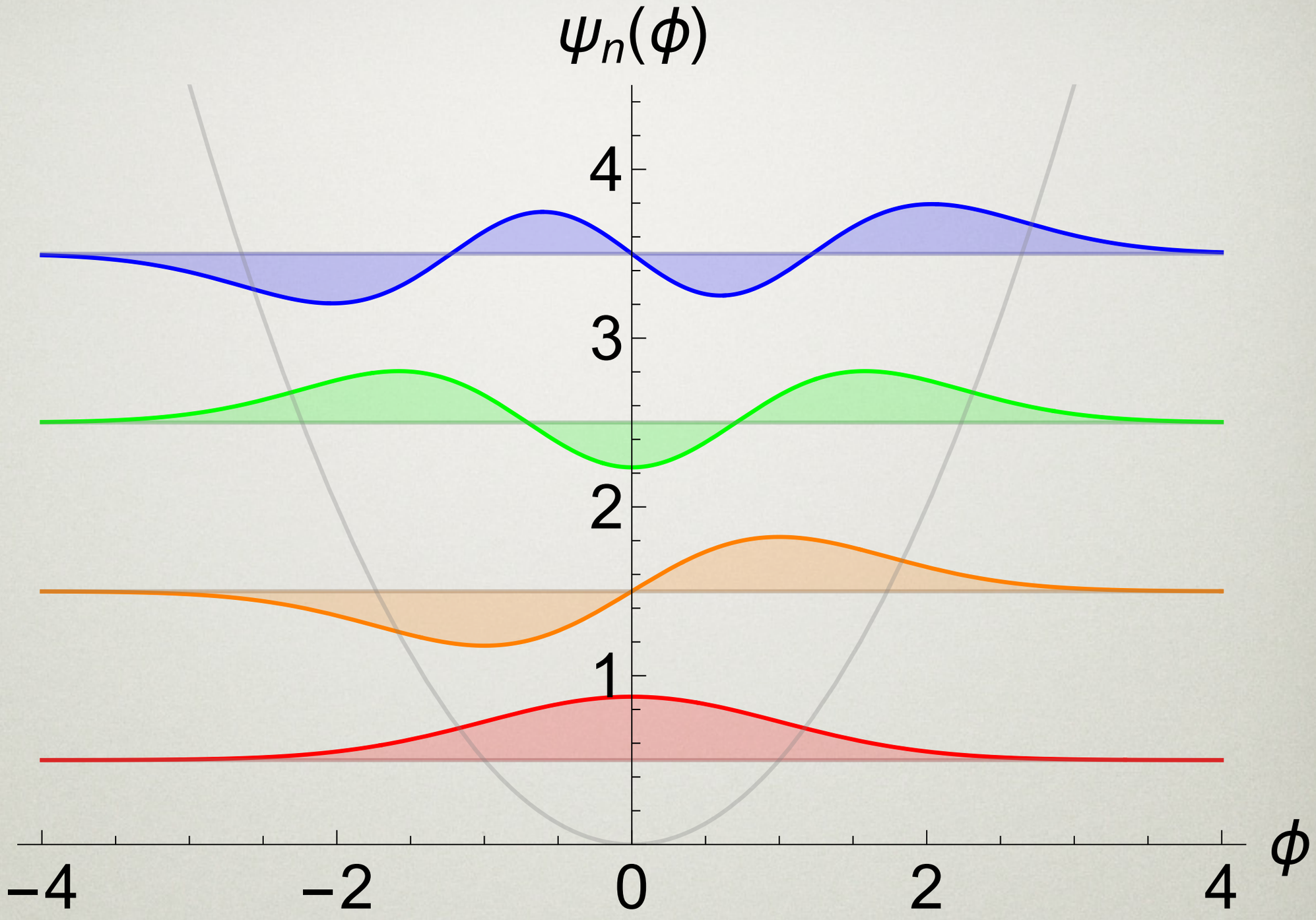


Figure source: Bluefors



An energy eigenstate is a superposition
of states where current is flowing in opposite directions!

QUANTUM FLUCTUATIONS

$$L = 1\,\mathrm{nH} \quad C = 10\,\mathrm{pF} \quad \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} \approx 1.6\,\mathrm{GHz}$$

typical values

$$A(t) = e^{i\mathcal{H}t/\hbar} A(0) e^{-i\mathcal{H}t/\hbar}$$

$$c(t) = e^{-i\omega_0 t} c$$

$$c^\dagger(t) = e^{i\omega_0 t} c^\dagger$$

$$\langle \phi(t)\phi(0) \rangle = \phi^2_{\mathrm{ZPF}} \left[\langle c^\dagger c \rangle e^{+i\omega_0 t} + \langle c c^\dagger \rangle e^{-i\omega_0 t} \right]$$

$$\langle c^\dagger c \rangle = \frac{1}{e^{\beta\hbar\omega_0} - 1} = \frac{1}{2}\left[\coth\left(\frac{\hbar\omega}{2k_B T} \right) - 1 \right] = n(\omega_0)$$

$$\langle c c^\dagger \rangle = n(\omega_0) + 1$$

$$\langle \phi(t)\phi(0) \rangle = \phi^2_{\mathrm{ZPF}} \left[\coth \frac{\beta\hbar\omega_0}{2} \cos\omega_0 t - i \sin\omega_o t \right]$$

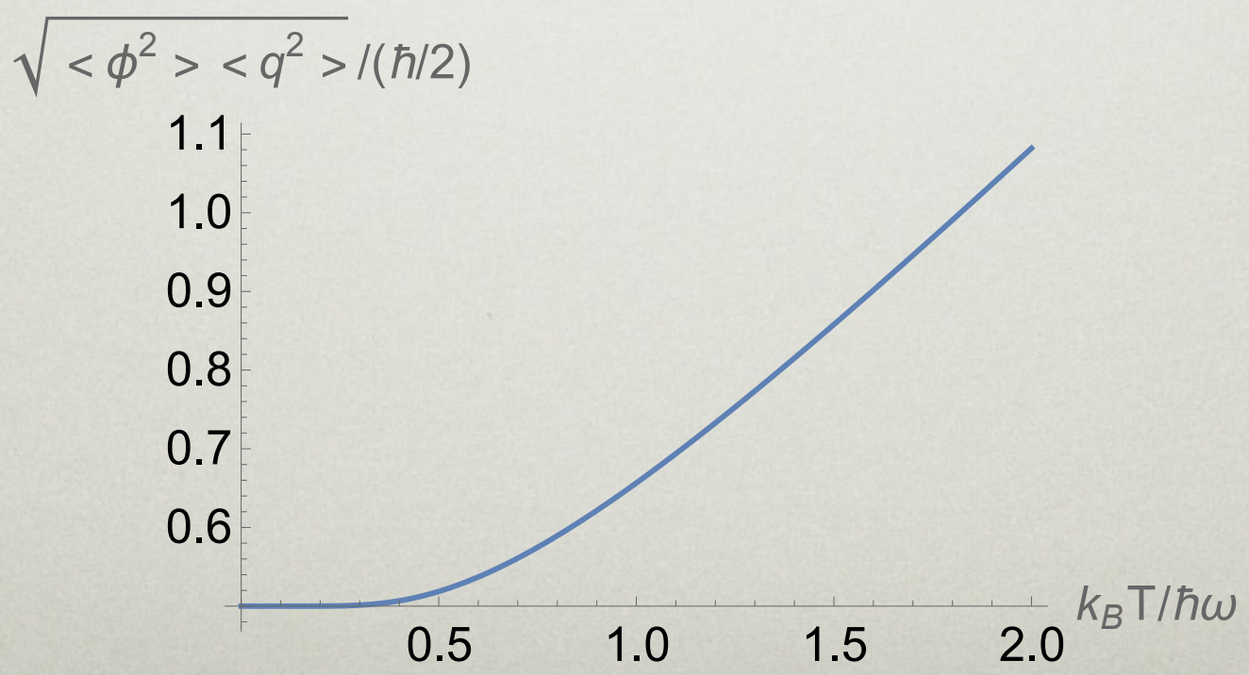
$$\langle \phi^2 \rangle = \frac{\hbar Z_0}{2} \coth \frac{\beta \hbar \omega_0}{2} \qquad \langle q^2 \rangle = \frac{\hbar}{2Z_0} \coth \frac{\beta \hbar \omega_0}{2}$$

$$\langle \phi^2 \rangle_{T \to 0} = \phi^2_{\mathrm{ZPF}} = \frac{\hbar Z_0}{2}$$

zero-temperature fluctuations controlled by impedance

$$\langle \phi^2 \rangle_{T \to \infty} = k_B T L$$

high-temperature fluctuations controlled by inductance alone

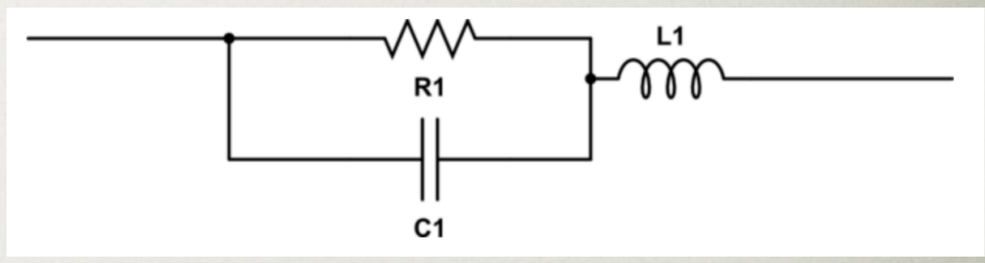


Dissipative quantum circuits

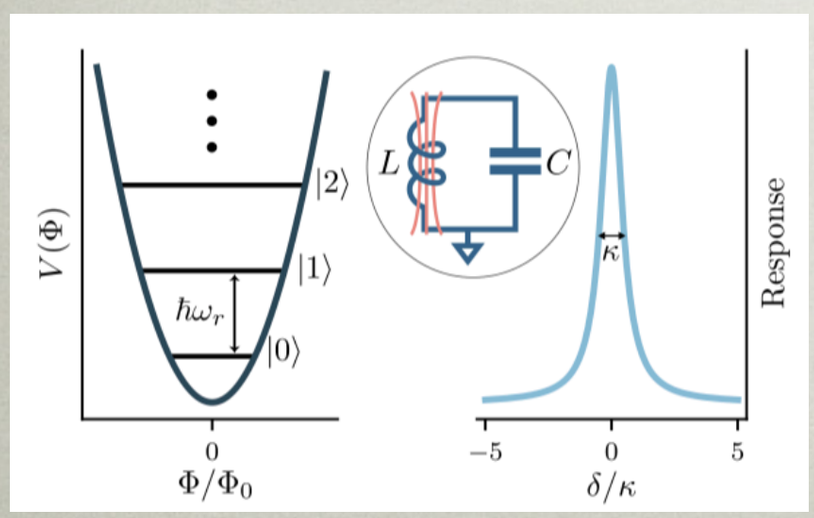
$$\langle \Phi(t)\Phi(0)\rangle = \frac{\hbar}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2}\left[\coth\frac{\beta\hbar\omega}{2}+1\right]\mathrm{Re}\left(Z[\omega]\right)\exp(-i\omega t)$$

quantum fluctuation-dissipation theorem

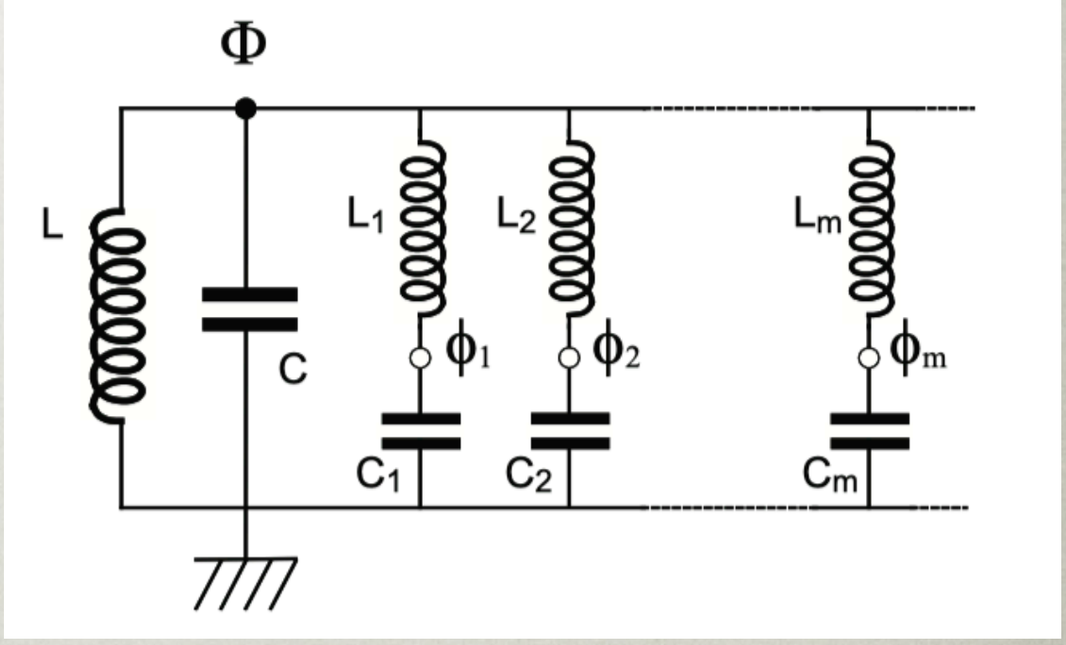
$$Y(\omega) = R^{-1}\frac{1}{1-i\frac{\omega}{\omega_c}} \qquad \omega_c \gg \omega_0$$



$$\kappa = \frac{1}{2RC\omega_0}$$ damping parameter

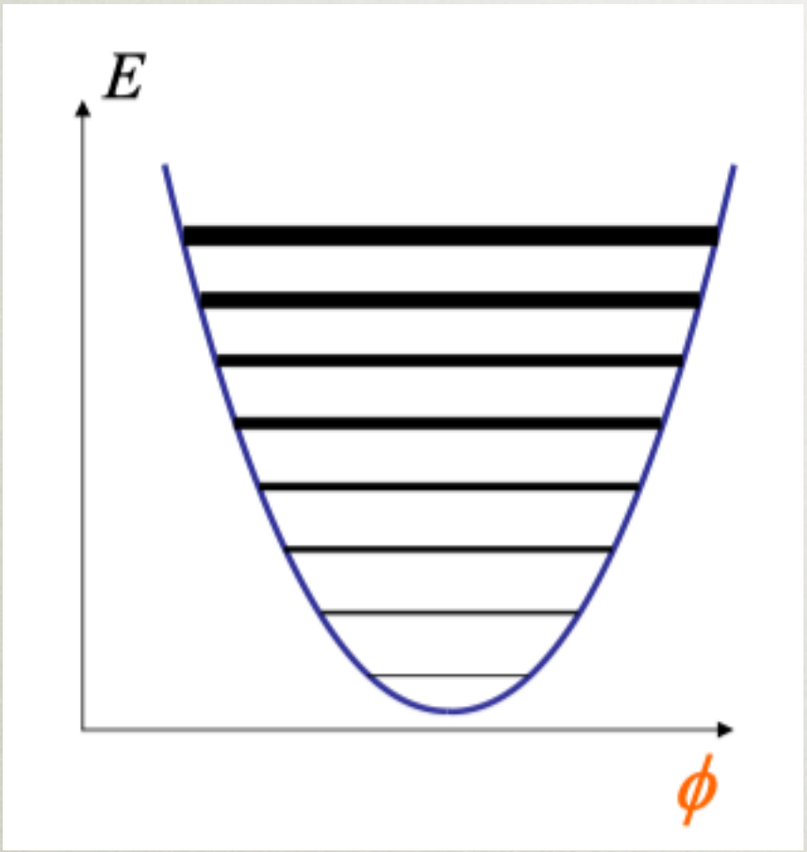


$$\delta = \omega - \omega_0$$



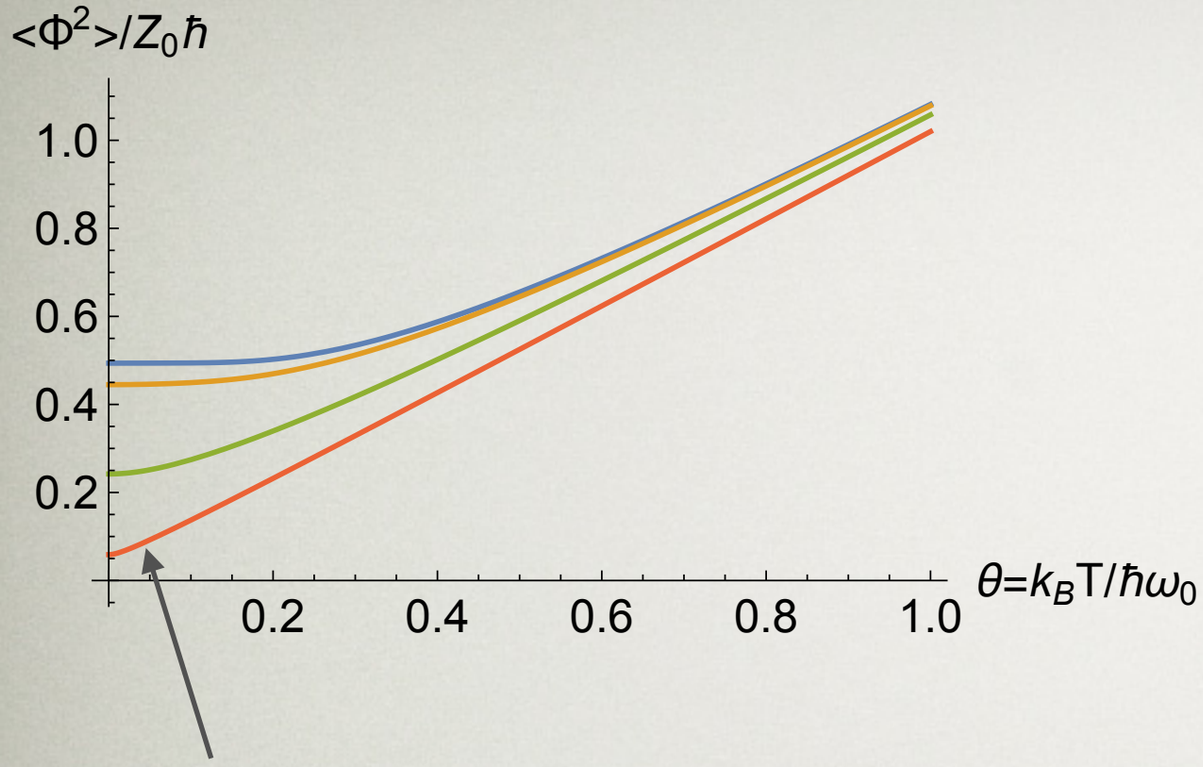
$$\frac{h}{(2e)^2} \approx 6.4\,\mathrm{k\Omega}$$ quantum of resistance

Figure source: Vool, Devoret



$$E_n = \hbar\omega_0 \left[n\left(1 + i\kappa\right) + \frac{1}{2} \right]$$

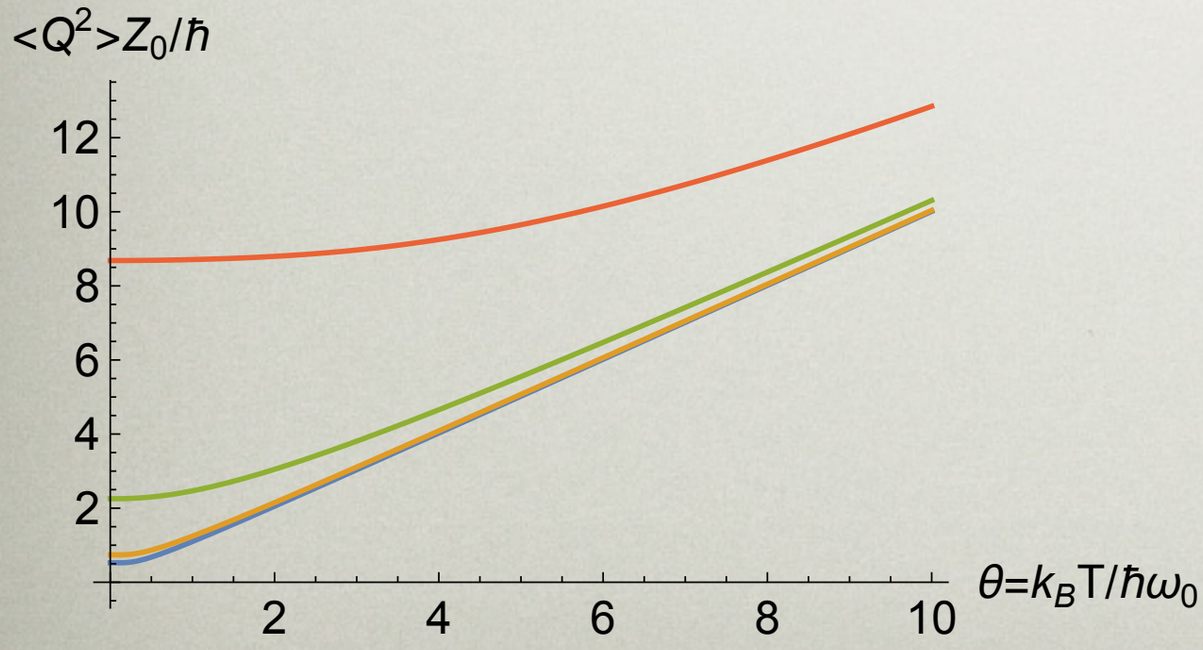
Figure source: Devoret lecture



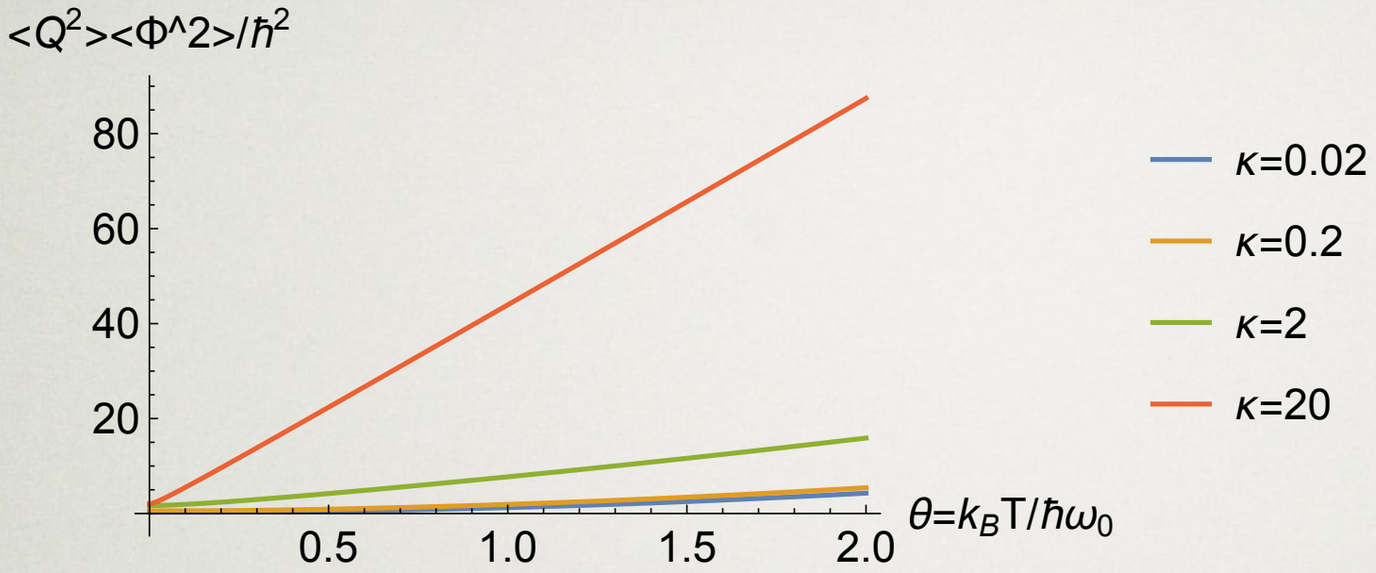
$$\langle \Phi^2 \rangle = \frac{\hbar Z_0}{2} \frac{2 \ln\left(\kappa + \sqrt{\kappa^2 + 1}\right)}{\pi \sqrt{\kappa^2 - 1}}$$

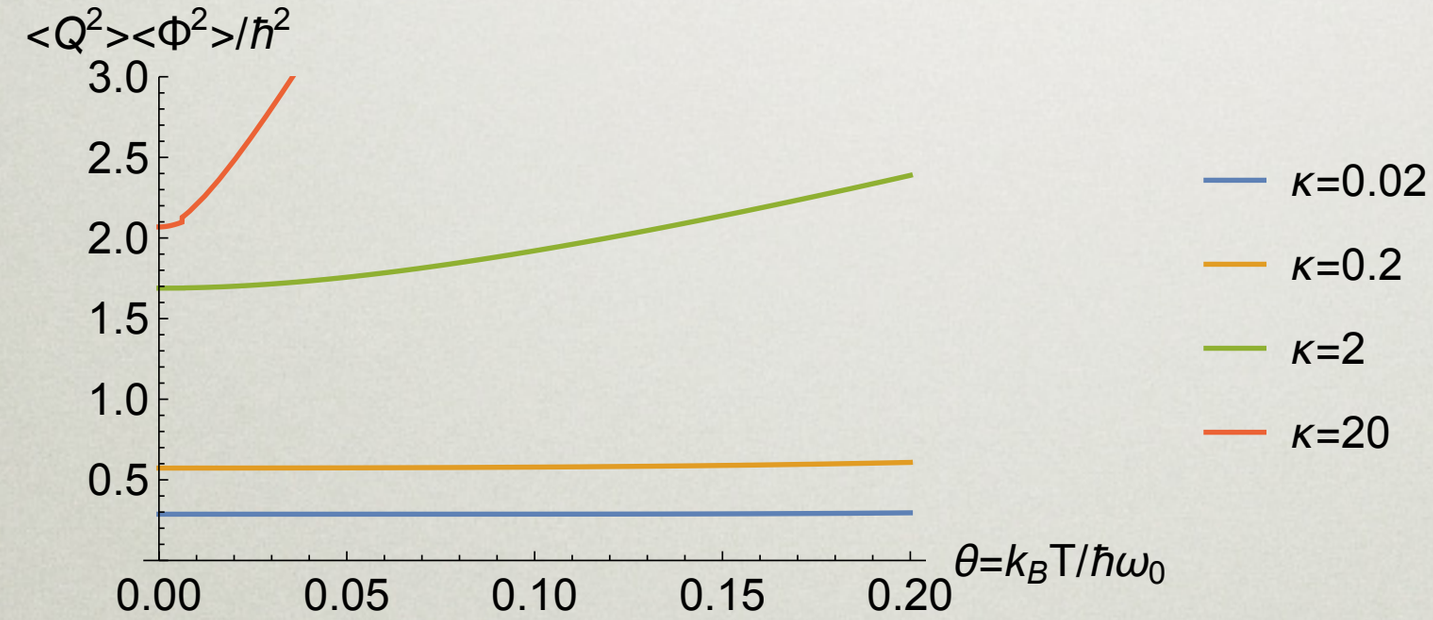
for T=0

zero-point fluctuations of the position of the particle in the well decrease



$$\omega_c/\omega_0 = 10$$





$$\omega_c/\omega_0 = 1000$$

If, therefore, adiabatic effects are "harmless" in the sense implied, it remains to discuss the effect of the non-adiabatic terms, i.e. these coupling the system to those modes of the environment which have frequency comparable to, or less than, these of the system ($\sim \omega_0$). As regards modes with frequencies far below ω_0, these are to all intents and purposes inert: they simply contribute to the universe wavefunction an overall multiplying factor which clearly has no role in destroying coherence. Thus the "dangerous" modes are those with frequencies of the order of ω_0. But these are precisely the modes which can exchange energy irreversibly with the system and hence act, *inter alia*, to damp its quasi-classical motion (we will confirm this in detail in the context of MQT and MQC in the next sections). Thus we reach a conclusion whose importance for our general program of research cannot be overestimated: *the part of the system-environment interaction which tends to destroy quantum coherence is just the part which causes dissipation in the classical motion.* Dissipation, and the destruction of coherence, are the classical and quantum sides of the same coin! (This point is discussed further in Leggett (1984a)).

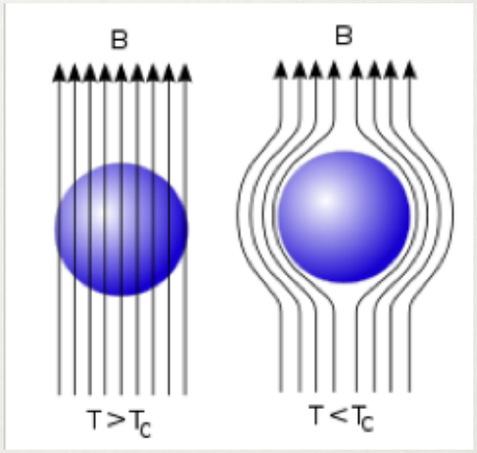
Leggett (1986)

Superconductivity

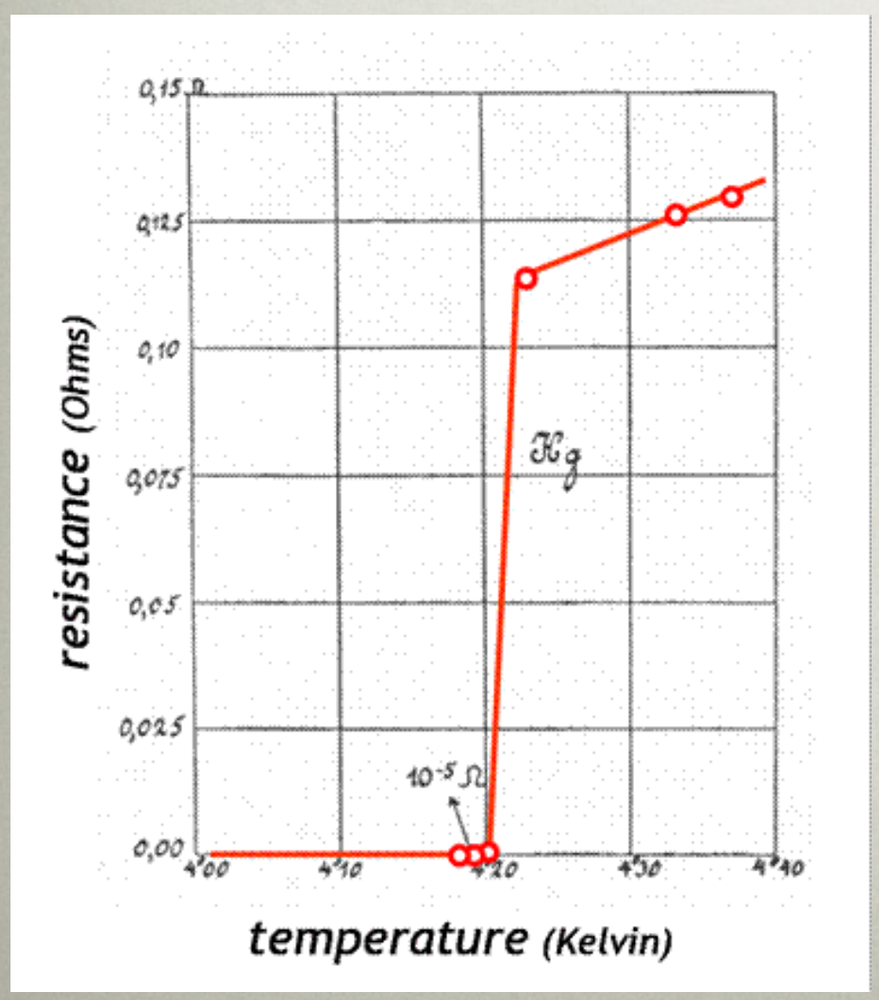


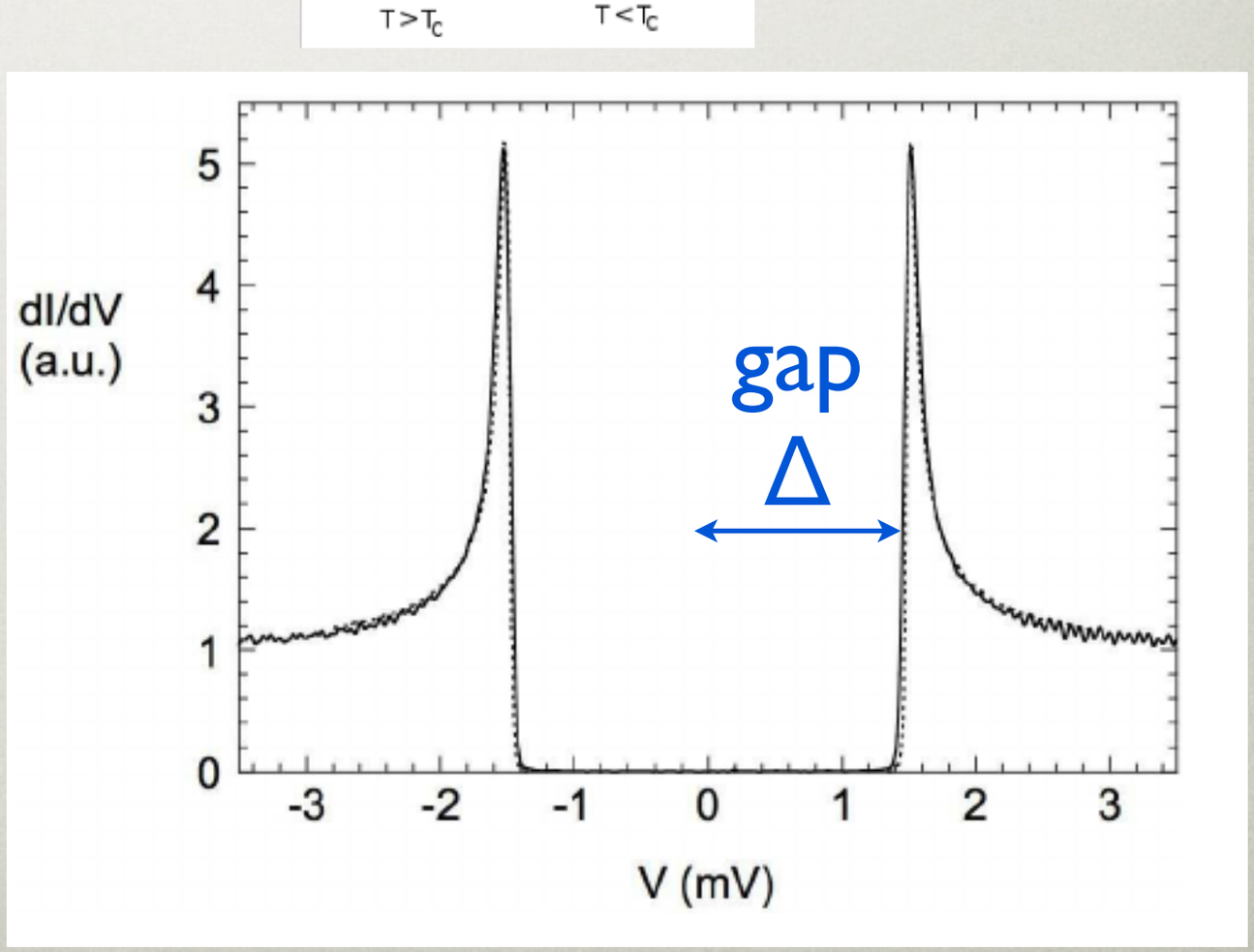
H. K. Onnes

(1911)



Meissner effect





Nb, T=170mK, Δ=1.5meV

H. Courtois, Grenoble
PRB (2005)

BCS theory in a nutshell

- attractive interaction between electrons due to phonons
- instability of the Fermi surface: condensation of Cooper pairs
- many Cooper pairs overlap in space
- mean-field theory is an excellent approximation

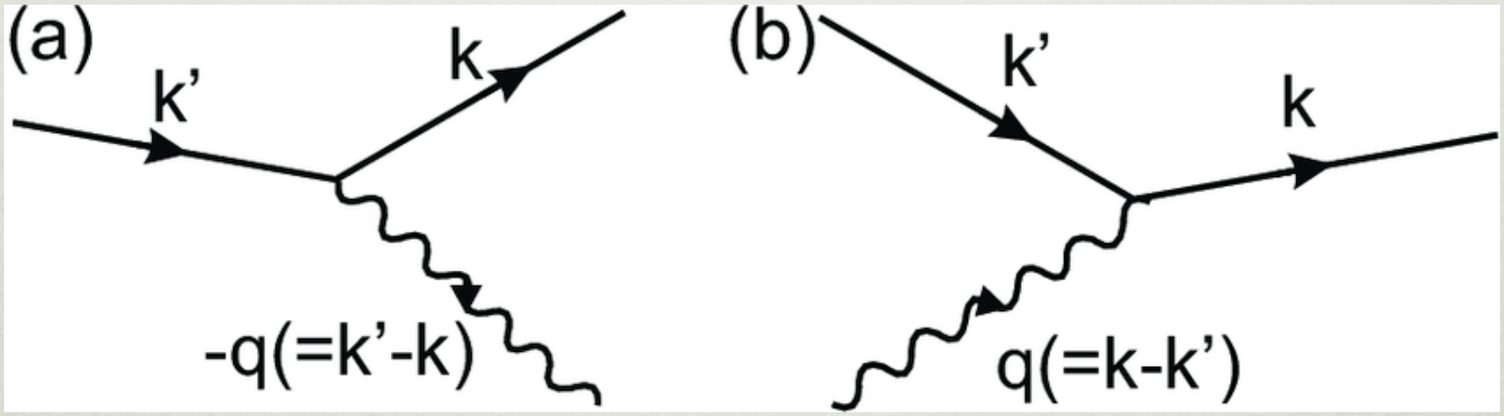
$$H_{\mathrm{BCS}} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_k \Delta \left(c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow} \right)$$

Bardeen, Cooper, Schrieffer (1957)

ELECTRON-PHONON INTERACTION

$$H = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_q \omega_q a^\dagger_q a_q + \sum_{k\sigma q} g_{kq} c^\dagger_{k+q,\sigma} c_{k,\sigma} \left(a_q + a^\dagger_{-q} \right)$$

(Fröhlich)



Effective e-e interaction

$$H_{\text{pairing}} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{k,q} V_{k,q} c^\dagger_{k+q,\uparrow} c^\dagger_{-k-q,\downarrow} c_{-k,\downarrow} c_{k,\uparrow}$$

$$V_{k,q} \approx \frac{|g_{kq}|^2 \hbar\omega_q}{(\epsilon_{k+q} - \epsilon_k)^2 - (\hbar\omega_q)^2}$$

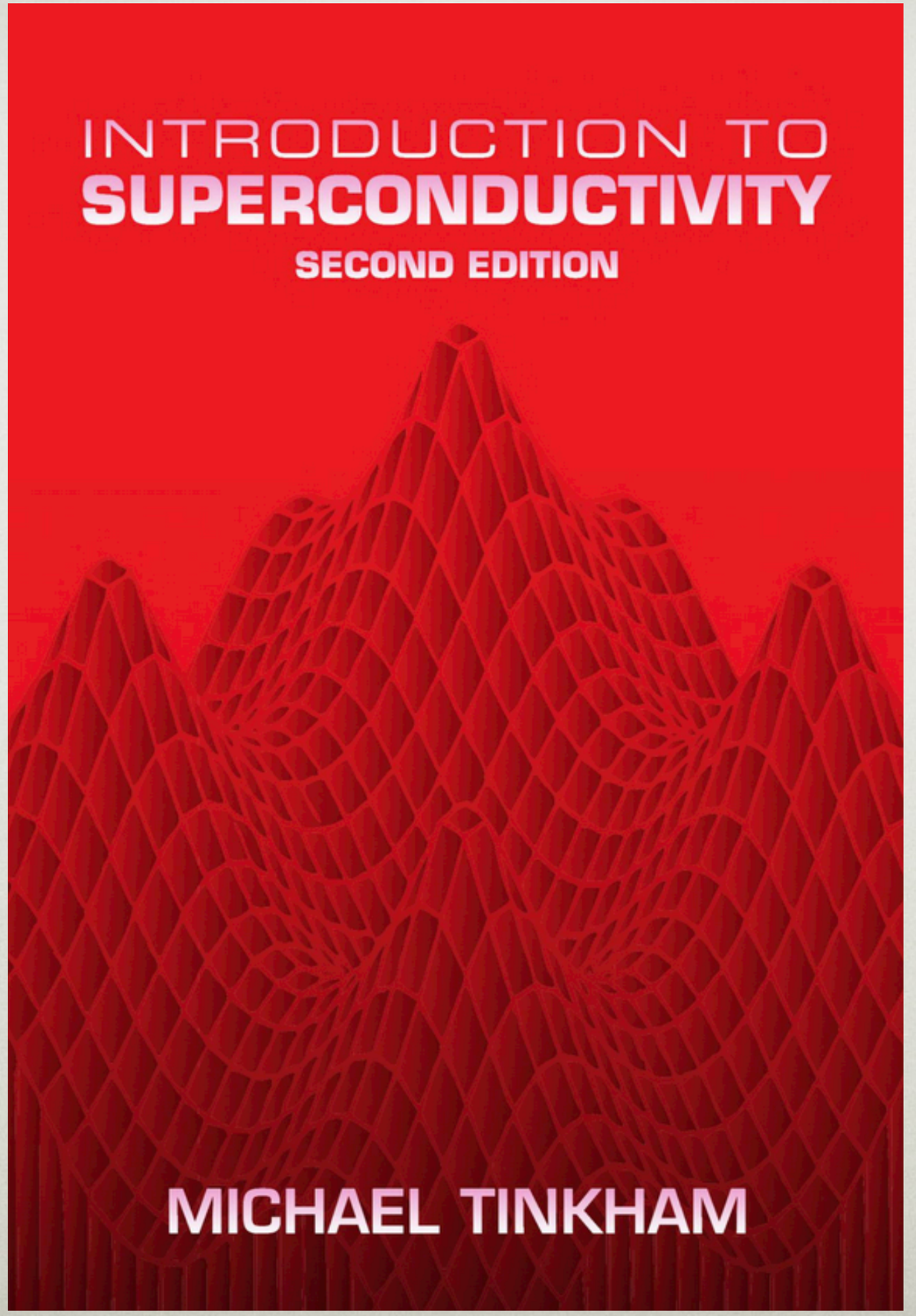
Attractive for

$$\epsilon_k, \epsilon_{k+q} \in [\epsilon_F - \hbar\omega_D : \epsilon_F + \hbar\omega_D]$$

$$H_{\text{red}} = \sum_{\mathrm{k}\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{k,k'} V_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

$$\Delta_k = \sum_{k'} V_{k,k'} \langle c_{-k'\downarrow} c_{k'\uparrow} \rangle$$

$$H_{\text{BCS}} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_{k} \Delta \left(c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow} \right)$$


INTRODUCTION TO
SUPERCONDUCTIVITY
SECOND EDITION
MICHAEL TINKHAM

chapter 3

Time-reversal invariance

$$\mathbf{k} \leftrightarrow -\mathbf{k}, \quad \uparrow \leftrightarrow \downarrow$$

Preserved in the presence of potential scattering.

Preserved in the presence of spin-orbit coupling.

Broken by spin-dependent scattering.

Broken by magnetic field.

Broken by current flow.

Anderson's theorem

Superconductivity in a conventional superconductor is robust with respect to (non-magnetic) disorder.

T_c barely depends on material purity (in fact, it increases upon increasing disorder!).

Anderson, P. W. (1959). "Theory of dirty superconductors". J. Phys. Chem. Solids. 11: 26–30.

Alternative derivation following Glazman (2021)

$$\mathcal{H} = \sum_{n,\sigma=\uparrow,\downarrow} \xi_n c^\dagger_{n\sigma} c_{n\sigma} + \sum_{klmn} \mathcal{H}_{klmn} c^\dagger_{k\uparrow} c^\dagger_{l\downarrow} c_{m\downarrow} c_{n\uparrow}$$

$$\mathcal{H}_{klmn} = \int \mathrm{d}\mathbf{r}_1 \mathrm{d}\mathbf{r}_2 V(\mathbf{r}_1 - \mathbf{r}_2) \psi^*_k(\mathbf{r}_1) \psi^*_l(\mathbf{r}_2) \psi_m(\mathbf{r}_2) \psi_n(\mathbf{r}_1)$$

$$V(\mathbf{r}) = \frac{\lambda}{\nu_0} \delta(\mathbf{r})$$ short-range (effective) interaction

$$\mathcal{H}_{kllk}, \mathcal{H}_{klkl}, \mathcal{H}_{kkmm}$$

Hartree, Fock, pairing terms

$$\mathcal{H}_{\mathrm{int}} = E_C (\hat{N} - \mathcal{N}_g)^2 + \lambda \delta \epsilon \hat{O}^\dagger \hat{O}$$

$$\hat{N} = \sum_{n,\sigma} c^\dagger_{n\sigma} c_{n\sigma} \qquad \hat{O} = \sum_{|\xi_n| < \hbar\omega_D} c_{n\downarrow} c_{n\uparrow}$$

analogy

$$\hat{O}^\dagger \hat{O} = \sum_{|\xi_n|<\hbar\omega_D} c^\dagger_{n\uparrow} c^\dagger_{n\downarrow} \sum_{|\xi_m|<\hbar\omega_D} c_{m\downarrow} c_{m\uparrow} \leftrightarrow \left(\sum_{|\xi_k|<\hbar\omega_D} c^\dagger_{\mathbf{k}\uparrow} c^\dagger_{-\mathbf{k}\downarrow} \right) \left(\sum_{|\xi_p|<\hbar\omega_D} c_{-\mathbf{p}\uparrow} c_{\mathbf{p}\downarrow} \right)$$

time-reversal conjugate pairs in both cases!

self-consistency equation

$$\Delta = (\lambda\delta\epsilon)\langle\hat{O}\rangle = (\lambda\delta\epsilon) \sum_{|\xi_m|<\hbar\omega_D} \langle c_{m\downarrow} c_{m\uparrow} \rangle$$

gap parameter

$$\mathcal{H}_{\rm BCS} = \sum_{n\sigma} \xi_n c^\dagger_{n\sigma} c_{n\sigma} + \Delta^* \sum_{|\xi_m|<\hbar\omega_D} c_{m\downarrow} c_{m\uparrow} + \Delta \sum_{|\xi_m|<\hbar\omega_D} c^\dagger_{m\uparrow} c^\dagger_{m\downarrow} - \frac{|\Delta|^2}{\lambda\delta\epsilon}$$

Bogoliubov transformation:

$$c_{n\uparrow} = u^*_n \gamma_{n\uparrow} + v_n \gamma^\dagger_{n\downarrow}$$
$$c^\dagger_{n\downarrow} = -v^*_n \gamma_{n\uparrow} + u_n \gamma^\dagger_{n\downarrow}$$

gauge fixing $\quad u_n = u^*_n, \;\; v_n = |v_n| e^{i\varphi} \;\rightarrow\; \Delta = |\Delta| e^{i\varphi}$

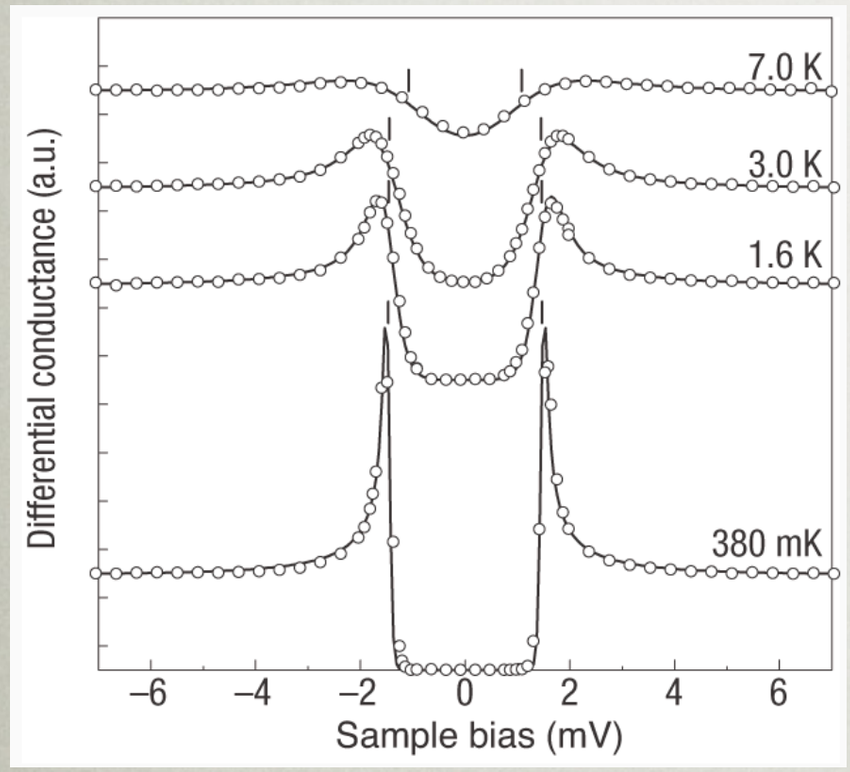
$$|v_n|^2 = 1 - |u_n|^2 = \frac{1}{2}\left(1 - \frac{\xi_n}{\epsilon_n}\right) \qquad \epsilon_n = \sqrt{\xi_n^2 + \Delta^2}$$

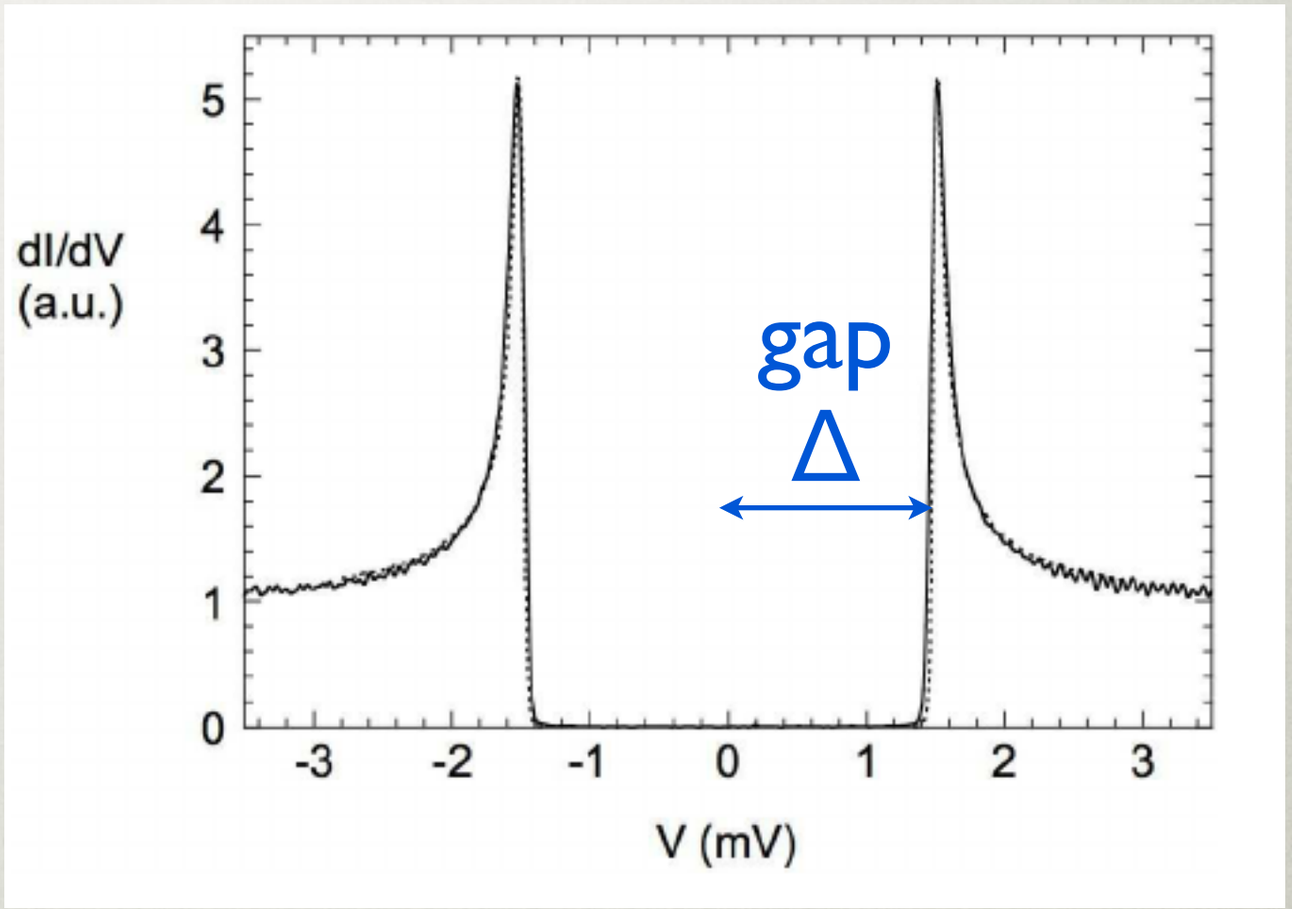
$$\mathcal{H}_{\rm qp} = \sum_{n,\sigma} \epsilon_n \gamma^\dagger_{n\sigma} \gamma_{n\sigma}$$

$$\Delta = (\lambda\delta\epsilon) \sum_n u^*_n v_n \left(1 - \langle \gamma^\dagger_{n\uparrow}\gamma_{n\uparrow}\rangle - \langle \gamma^\dagger_{n\downarrow}\gamma_{n\downarrow}\rangle\right)$$

T=0 $\qquad |\Delta| \approx 2\hbar\omega_d \exp\left(-\frac{1}{\lambda}\right)$ $\qquad$ valid for $\;\; \delta\epsilon \ll |\Delta| \ll \hbar\omega_D$

Superconducting gap





Nb, T=170mK, Δ=1.5meV

H. Courtois, Grenoble
PRB (2005)

scanning tunneling microscope (STM) dI/dV spectra
(proportional to the local **density of states**)

Bogoliubov quasiparticles

$$H_{\mathrm{BCS}} = \sum_{k\sigma} \epsilon_k c^\dagger_{k\sigma} c_{k\sigma} + \sum_k \Delta \left(c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} + c_{-k\downarrow} c_{k\uparrow} \right)$$

$$\gamma_{k\uparrow} = u_k c_{k\uparrow} - v_k c^\dagger_{-k\downarrow}$$

$$u_k^2 = \frac{1}{2}\left(1 + \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta^2}}\right)$$

$$v_k^2 = \frac{1}{2}\left(1 - \frac{\epsilon_k}{\sqrt{\epsilon_k^2 + \Delta^2}}\right)$$

many-particle picture

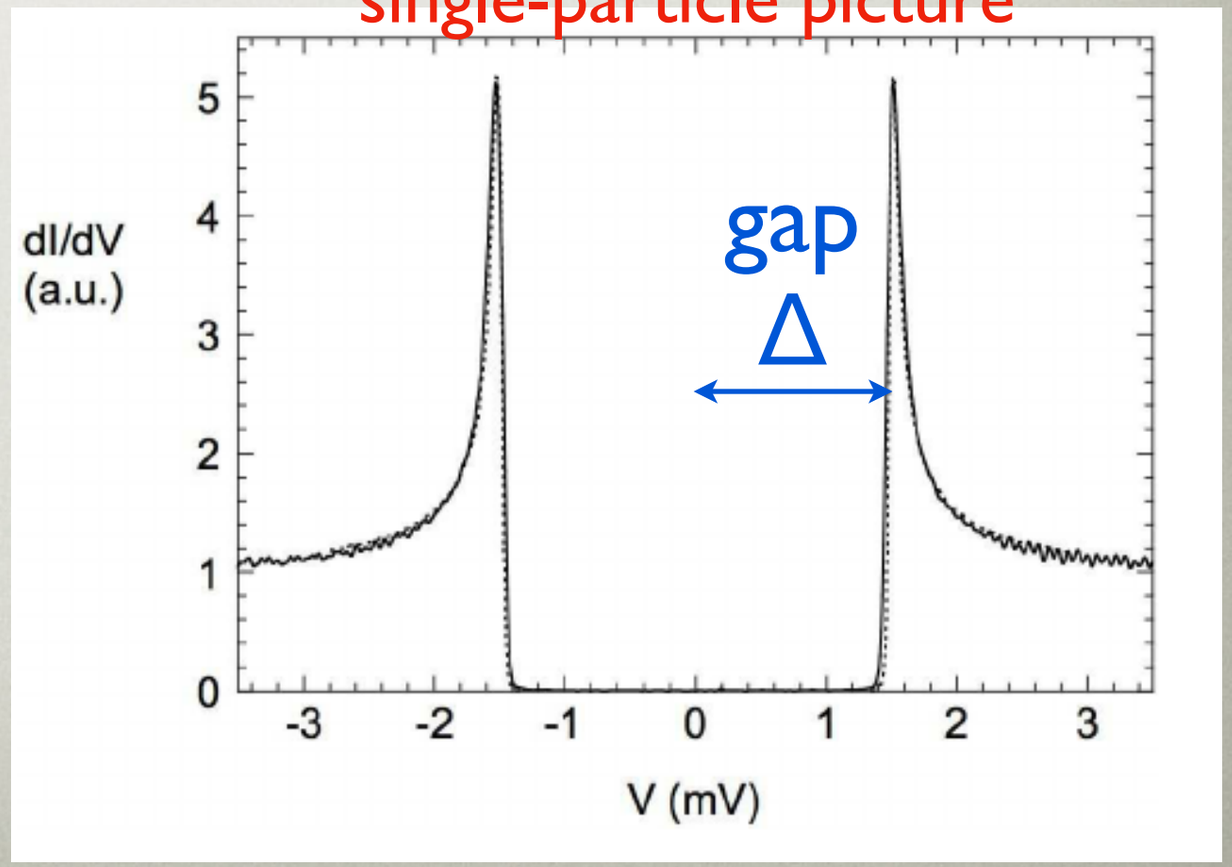
$$E_k = \sqrt{\epsilon_k^2 + \Delta^2}$$

quasiparticle continuum



ground state

single-particle picture



$$|\psi_\varphi\rangle = \prod_n \left(u_n + v_n c^\dagger_{n\uparrow} c^\dagger_{n\downarrow}\right)|0\rangle$$

$$u_n = u_n^*,\ \ v_n = |v_n| e^{i\varphi} \ \rightarrow\ \Delta = |\Delta| e^{i\varphi}$$

$$|\psi_N\rangle = \int_0^{2\pi} \mathrm{d}\varphi\, e^{-iN\varphi} |\psi_\varphi\rangle$$

N: number of Cooper pairs

$$\begin{aligned} N\,|\psi_N\rangle &= N \int_0^{2\pi} \mathrm{d}\varphi e^{-iN\varphi}\,|\psi_\varphi\rangle \\ &= \int_0^{2\pi} \mathrm{d}\varphi \left(i\frac{\mathrm{d}}{\mathrm{d}\varphi} e^{-iN\varphi}\right)|\psi_\varphi\rangle \\ &= \int_0^{2\pi} \mathrm{d}\varphi e^{-iN\varphi} \left(-i\frac{\mathrm{d}}{\mathrm{d}\varphi}\,|\psi_\varphi\rangle\right) \end{aligned}$$

$$\hat{T} = \sum_N |N+1\rangle\langle N|$$

$$|\psi_{N+1}\rangle = \int_0^{2\pi} \mathrm{d}\varphi e^{-i(N+1)\varphi}\,|\psi_\varphi\rangle = \int_0^{2\pi} e^{-i\varphi}\left(e^{-iN\varphi}\,|\psi_\varphi\rangle\right)$$

$$\hat{N} = -i\frac{\mathrm{d}}{\mathrm{d}\varphi}$$

$$\hat{T} = e^{-i\varphi}$$

$$[\hat{N}, e^{-i\varphi}] = e^{-i\hat{\varphi}}$$

N integer, ϕ compact

Richardson model

charge-conserving Hamiltonian for superconducting grains

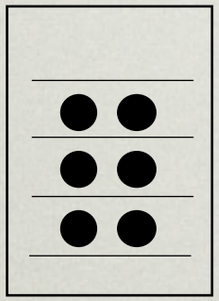
$$H = \sum_i \epsilon_i c^\dagger_{i\sigma} c_{i\sigma} - \alpha d \sum_{i,j} c^\dagger_{i\uparrow} c^\dagger_{i\downarrow} c_{j\downarrow} c_{j\uparrow}$$

coupling strength

interlevel spacing

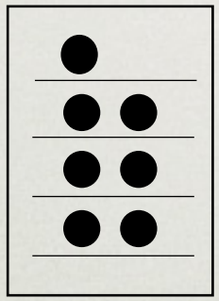
coupling between the time-reversal conjugate pairs!

even N:

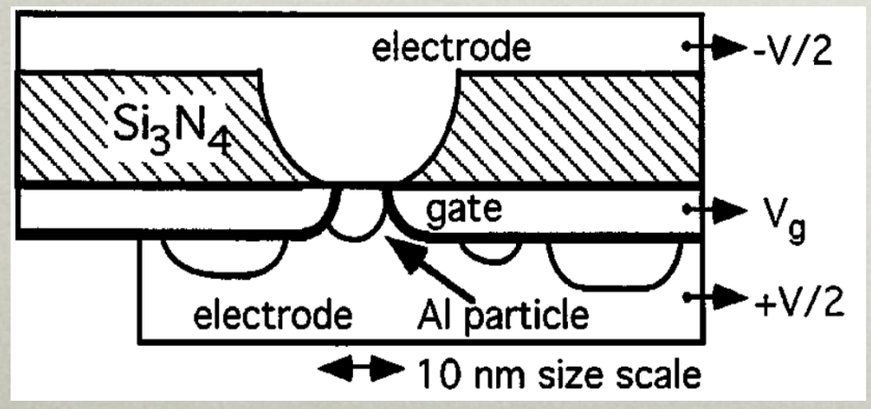


$S = 0$

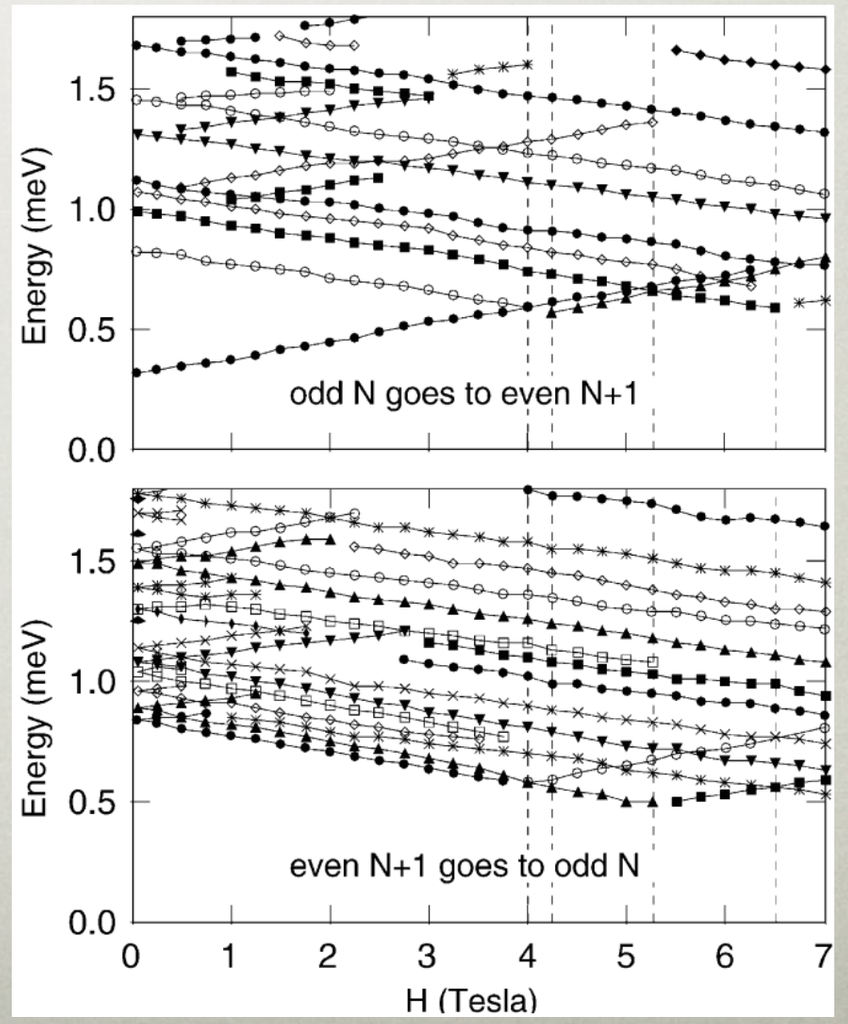
odd N:



$S = 1/2$

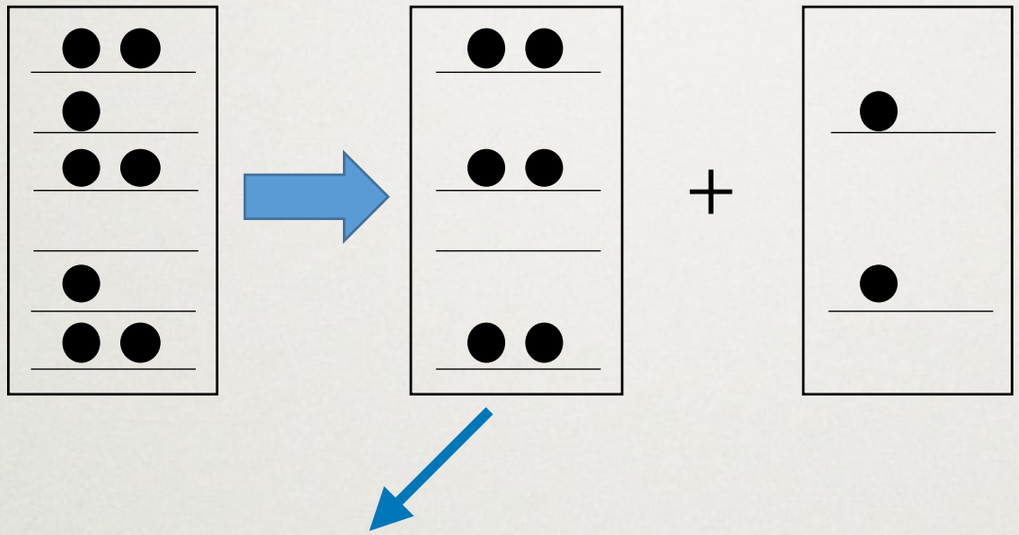


Ralph, Black, Tinkham, 1996, 1997





Exact solution of Richardson model



single-occupied levels are inert
(blocking effect!)

$$H = \sum_{ij} (2\epsilon_i \delta_{ij} - \alpha d) b_i^\dagger b_j \qquad b_i^\dagger = c_{i\uparrow}^\dagger c_{i\downarrow}^\dagger$$

hard-core bosons

$$|\psi\rangle = \prod_\alpha \left(\sum_{i=1}^{N} \frac{b_i^\dagger}{2\epsilon_i - E_\alpha} \right) |0\rangle$$

eigenvalue equation:

$$\frac{1}{\alpha d} - \sum_j \frac{1}{2\epsilon_j - E_\nu} + \sum_{\mu \neq \nu} \frac{2}{E_\mu - E_\nu} = 0$$

Fully equivalent to BCS in the large-N, small-α limit

Richardson, 1963

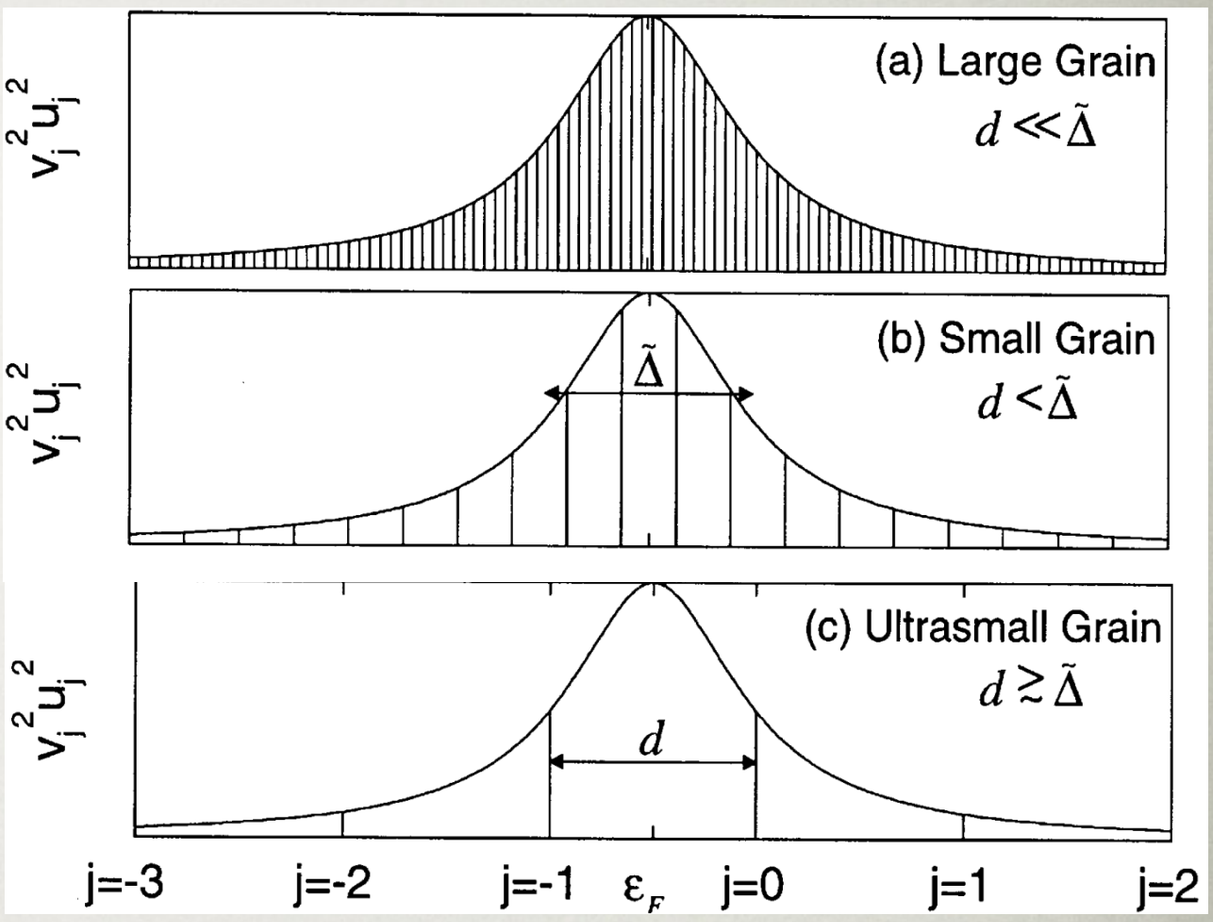
What is the smallest system size at which the superconductivity persists?

interlevel spacing $$d \propto \frac{1}{V}$$

number of Cooper pairs = d/Δ

$$\frac{d}{\Delta} \approx 1$$

Anderson (1959)



von Delft, Ralph, 2001

Averin, Nazarov (1992)
Jankó, Smith, Ambegaokar (1994)
Golubev, Zaikin (1996)

von Delft, Zaikin, Golubev, Tichy (1996)
Mastellone, Falci, Fazio (1998)
Braun, von Delft (1998,1999)
Dukelsky, Sierra (1999,2000)

Tuominen, Hergenrother, Tighe, Tinkham (1992)
Lafarge, Joyez, Esteve, Urbina, Devoret (1993)
Eiles, Martinis, Devoret (1993)

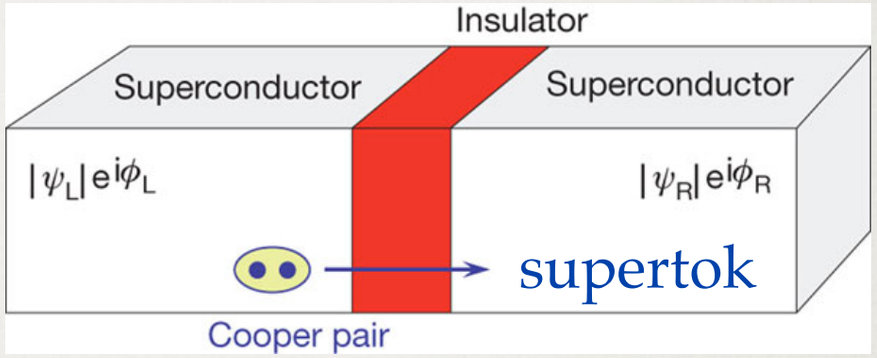


Brian Josephson

theory: 1962
(aged 22)

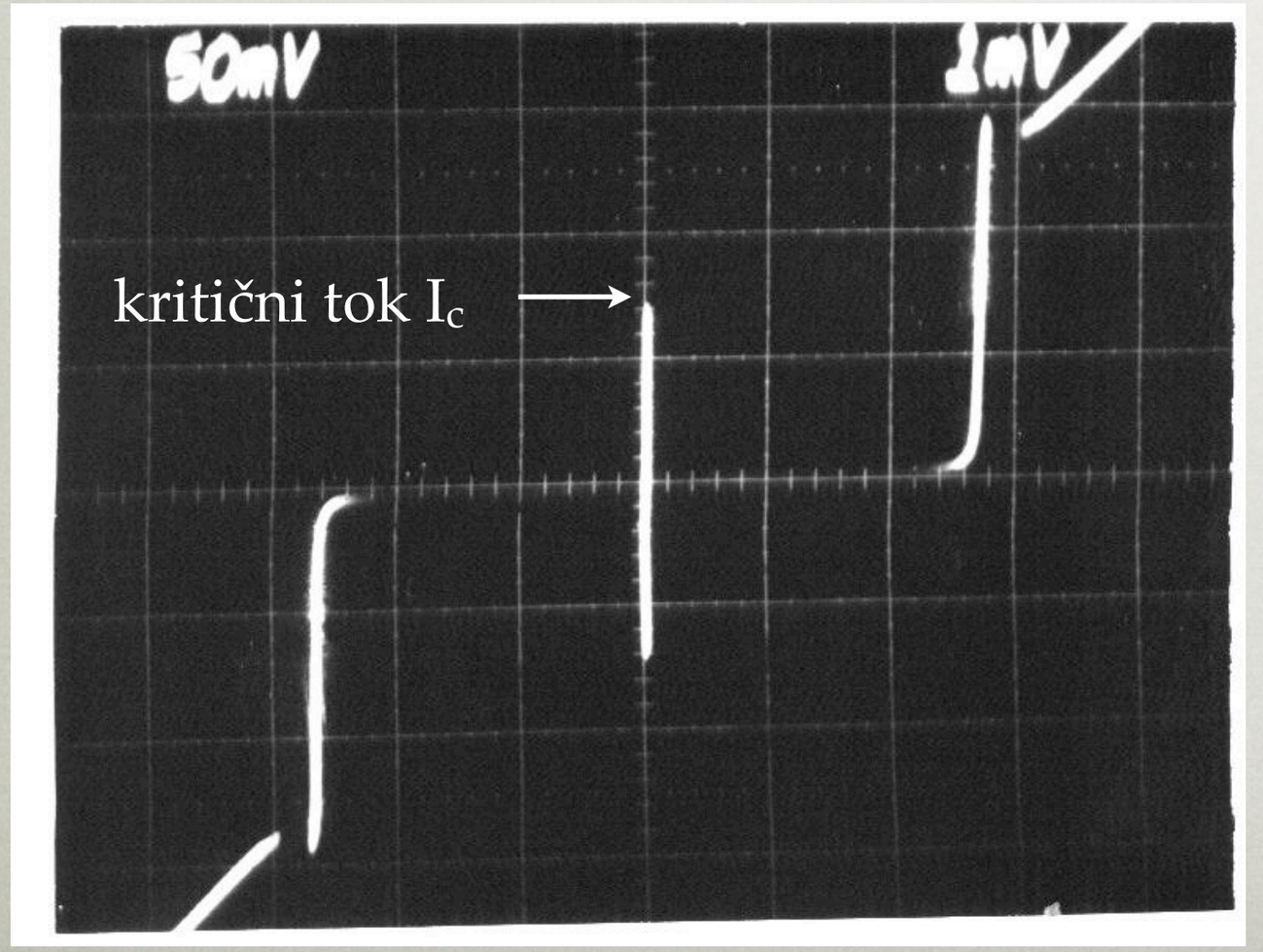


1973



Vir slike: Wikipedia (CC BY, avtor: B.J.)

$$I(t) = I_c \sin[\Delta\phi(t)] \qquad U(t) = \frac{\hbar}{2e}\frac{\mathrm{d}\Delta\phi(t)}{\mathrm{d}t}$$

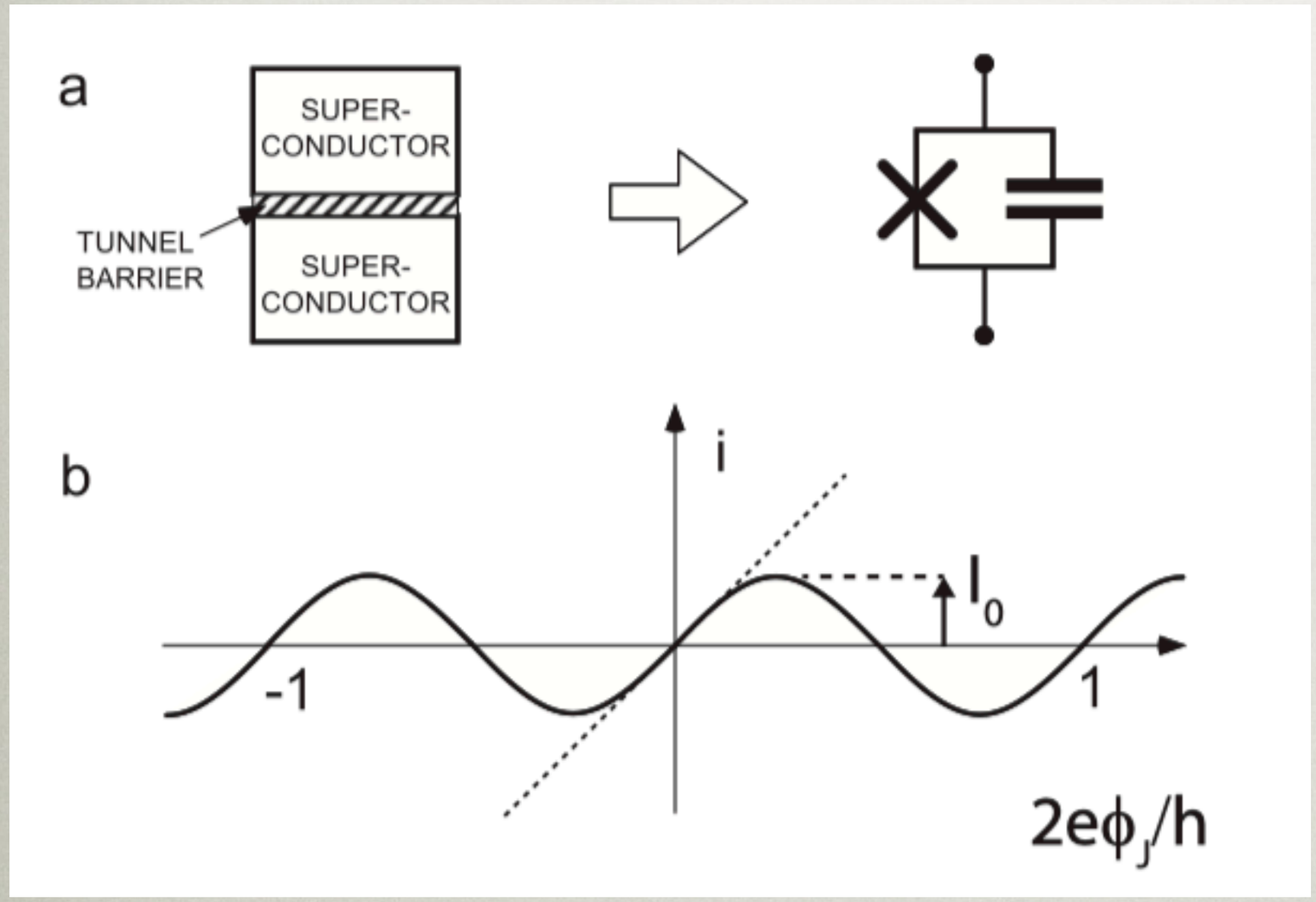


current

voltage

Vir slike: Wikipedia, Josephson current (javna last)

Josephson element



$$i(t) = I_0 \sin\left\{\frac{2e}{\hbar}[\Phi(t) - \Phi_{\text{offset}}]\right\}$$

$$\mathcal{E}(t) = \frac{\phi_0^2}{L_J}\left\{1 - \cos\left[\frac{\Phi(t) - \Phi_{\text{offset}}}{\phi_0}\right]\right\}$$

$$\varphi = \frac{2e}{\hbar}\phi \mod 2\pi$$

Figure source: Vool, Devoret

Energy of a Josephson junction

$$h_J(\phi) = E_J\left[1 - \cos\frac{\phi - \phi_{\text{offset}}}{\phi_0}\right] \qquad \varphi = \frac{\phi - \phi_{\text{offset}}}{\phi_0} \qquad \phi_0 = \frac{\hbar}{2e}$$ flux quantum

$$E_J(1 - \cos\varphi) = \frac{E_J}{2}\varphi^2 - \frac{E_J}{24}\varphi^4 + \mathcal{O}(\varphi^6)$$

$$E_J = \frac{1}{8}\frac{h}{e^2}G_t\Delta$$ Ambegaokar-Baratoff relation

normal-state conductance of the junction

VOLUME 10, NUMBER 11 PHYSICAL REVIEW LETTERS 1 JUNE 1963

TUNNELING BETWEEN SUPERCONDUCTORS*

Vinay Ambegaokar and Alexis Baratoff†
Department of Physics and Laboratory of Atomic and Solid State Physics, Cornell University, Ithaca, New York
(Received 22 April 1963)

In a recent note, Josephson[1] has predicted some new effects in the tunneling characteristics of a system of two superconductors separated by an oxide layer. The effects are an oscillating supercurrent of frequency $2eV/h$ (where V is the applied voltage), and a possible dc supercurrent at zero voltage.[2] In this Letter we reproduce and somewhat generalize Josephson's calculation by another method, the method of the thermodynamic Green's functions, which brings out the assumptions that

equal energy is[3]

$$T_{kq} = \hbar\int\{(\hbar/2mi)[\varphi_k^*(\vec{x})\partial\chi_q(\vec{x})/\partial z - \chi_q(\vec{x})\partial\varphi_k^*(\vec{x})/\partial z] - (e/mc)A_z\varphi_k^*\chi_q\}_{z=z_0}dxdy. \quad (3)$$

Above $\varphi_k(\vec{x})$ and $\chi_q(\vec{x})$ are the wave functions for the states with quantum numbers k and q, the z

Phase difference operator

$$N = N_1 - N_2 \qquad \hat{N} = \sum_N N \left|N\right\rangle \left\langle N\right| \qquad \theta = \varphi_1 - \varphi_2$$

$$\hat{h}_{\mathrm{cpt}} = -\frac{E_J}{2} \sum_{N=-\infty}^{\infty} \left[\left|N\right\rangle \left\langle N+1\right| + \left|N+1\right\rangle \left\langle N\right|\right]$$

$$\left|\theta\right\rangle = \sum_{N=-\infty}^{\infty} e^{iN\theta} \left|N\right\rangle$$

$$\left|N\right\rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, e^{-iN\theta} \left|\theta\right\rangle$$

$$e^{i\hat{\theta}} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, e^{i\theta} \left|\theta\right\rangle \left\langle\theta\right|$$

$$e^{i\hat{\theta}} \left|N\right\rangle = \left|N-1\right\rangle$$

$$\hat{h}_{\mathrm{cpt}} = -\frac{E_J}{2} \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta \, \left(e^{i\theta} + e^{-i\theta}\right) \left|\theta\right\rangle \left\langle\theta\right|$$

$$\hat{h}_{\mathrm{cpt}} = -E_J \cos\hat{\theta}$$

energy cost to transfer one Cooper pair: (2e)v

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta} = \frac{1}{i\hbar}[\hat{\theta}, \mathcal{H}] = \frac{1}{\hbar}\frac{\partial}{\partial \hat{N}}\mathcal{H}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\theta} = \frac{2e}{\hbar}\hat{v}$$

$$\hat{i} = 2e\frac{\mathrm{d}\hat{N}(t)}{\mathrm{d}t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{N} = \frac{1}{i\hbar}[\hat{N}, \mathcal{H}] = -\frac{1}{\hbar}\frac{\partial}{\partial \hat{\theta}}\mathcal{H}$$

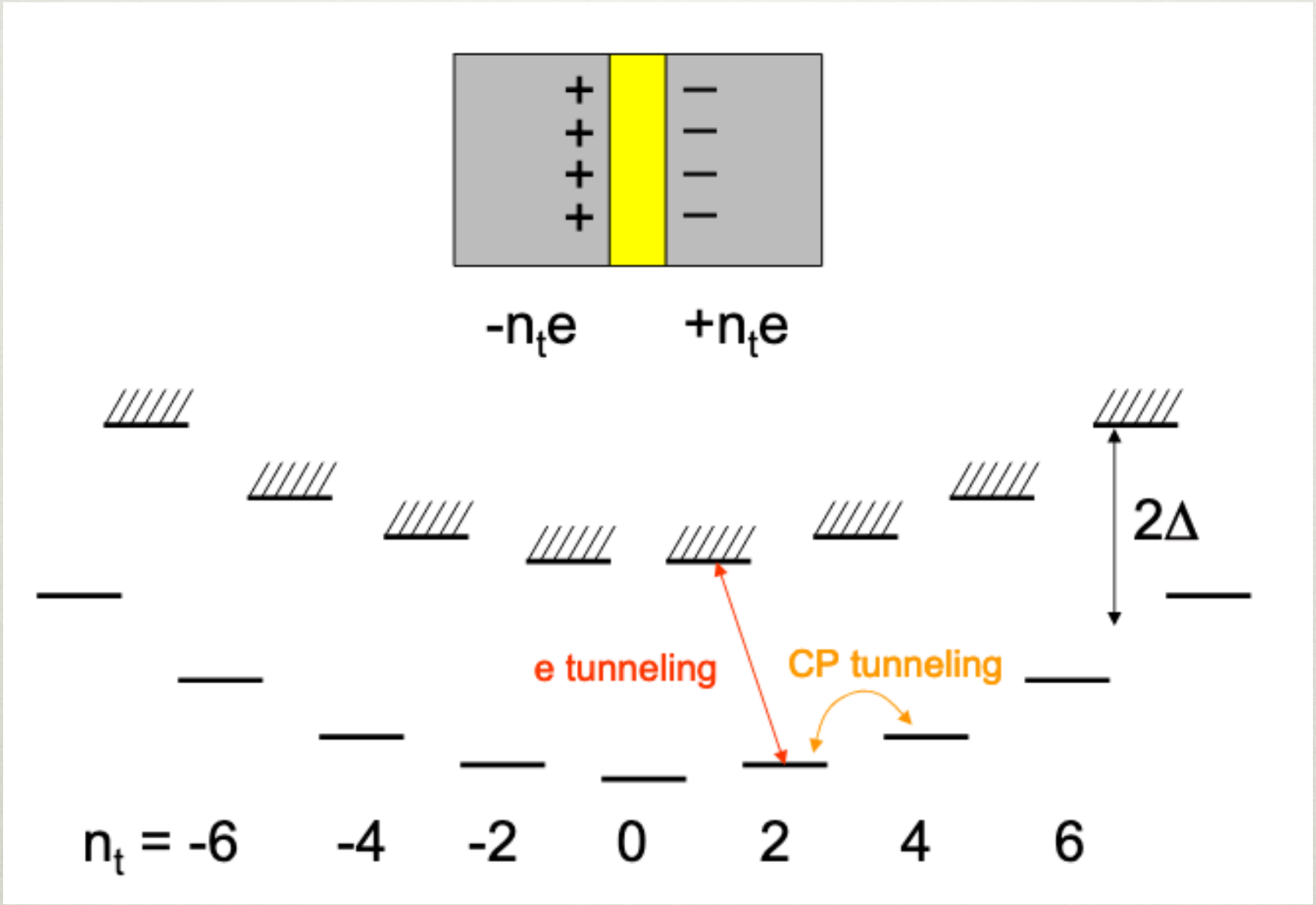
$$\hat{i} = I_0 \sin\hat{\theta}$$

$$I_0 = \frac{2e}{\hbar}E_J$$

Reminder: Hamilton's equations

$$\dot{\phi}_n = \frac{\partial \mathcal{H}}{\partial q_n} \qquad \dot{q}_n = -\frac{\partial \mathcal{H}}{\partial \phi_n}$$

SINGLE-ELECTRON VS. COOPER-PAIR TUNNELING



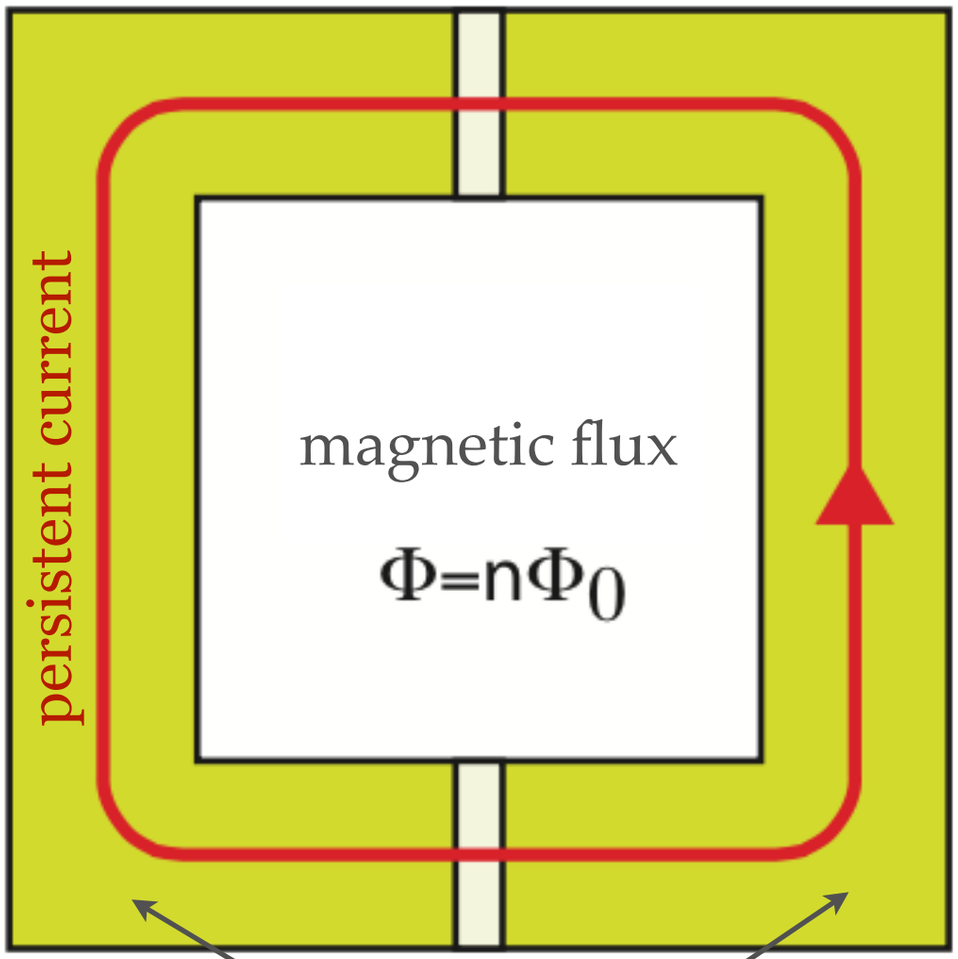
All real Josephson junctions have finite capacitance!

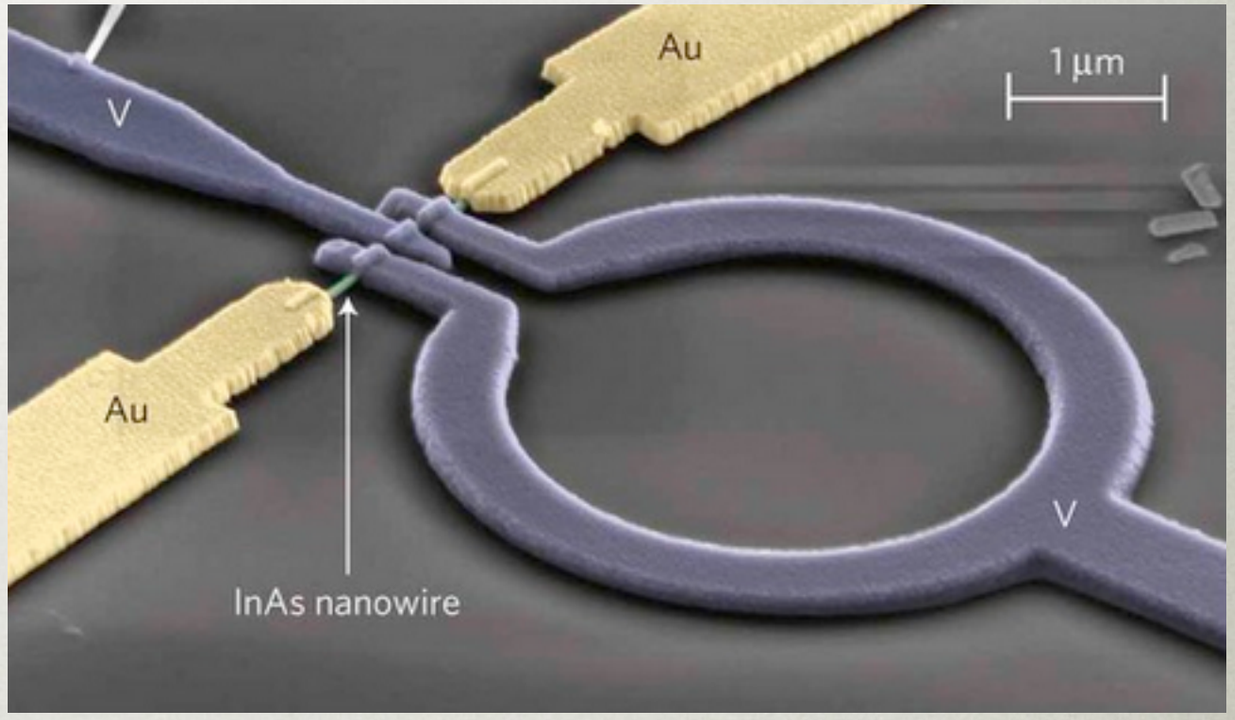
$E_c \hat{n}^2$ electrostatic term

cf. the notion of quasiparticle poisoning

Figure source: Devoret lectures

Superconducting quantum interference device (SQUID)

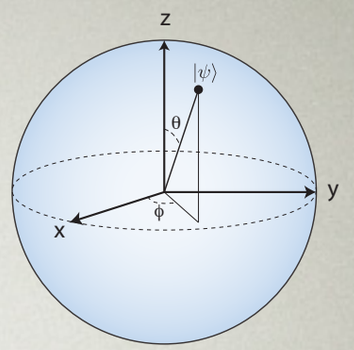




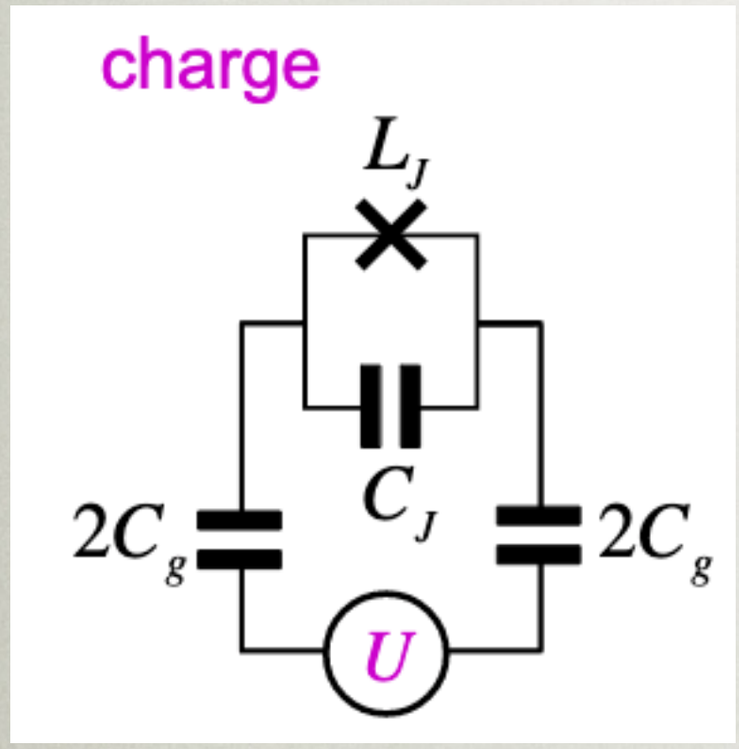
Giazotto et al. (2011)

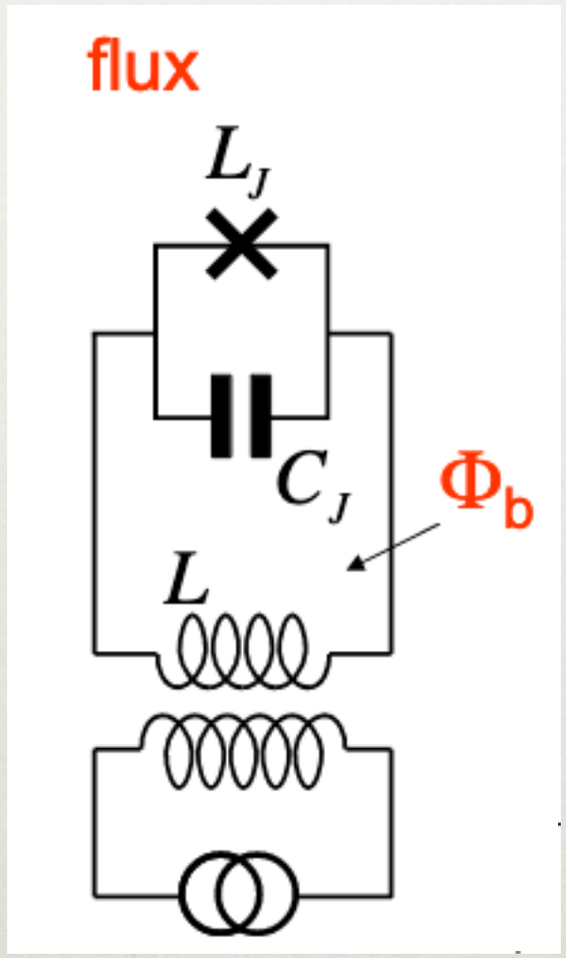
$$\Phi = B \times S$$

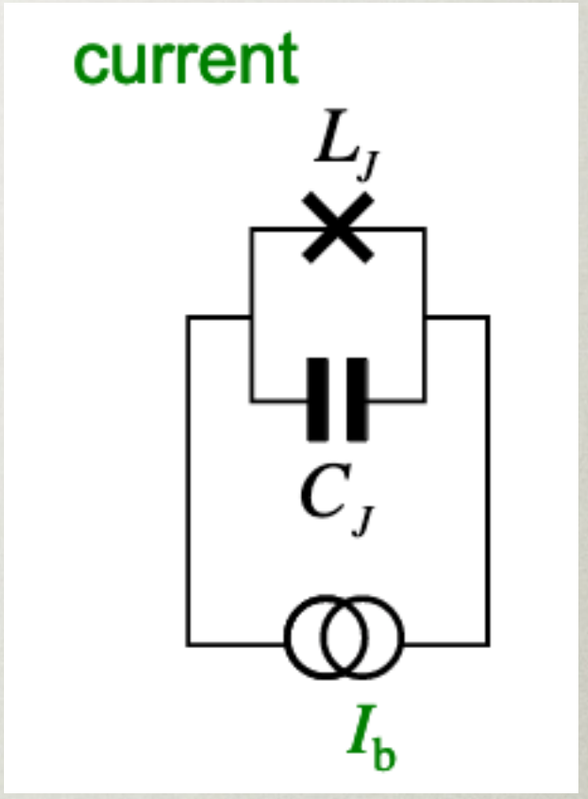
$$\Phi_0 = \frac{h}{2e}$$ flux quantum



Superconducting qubits







q: Cooper pair number

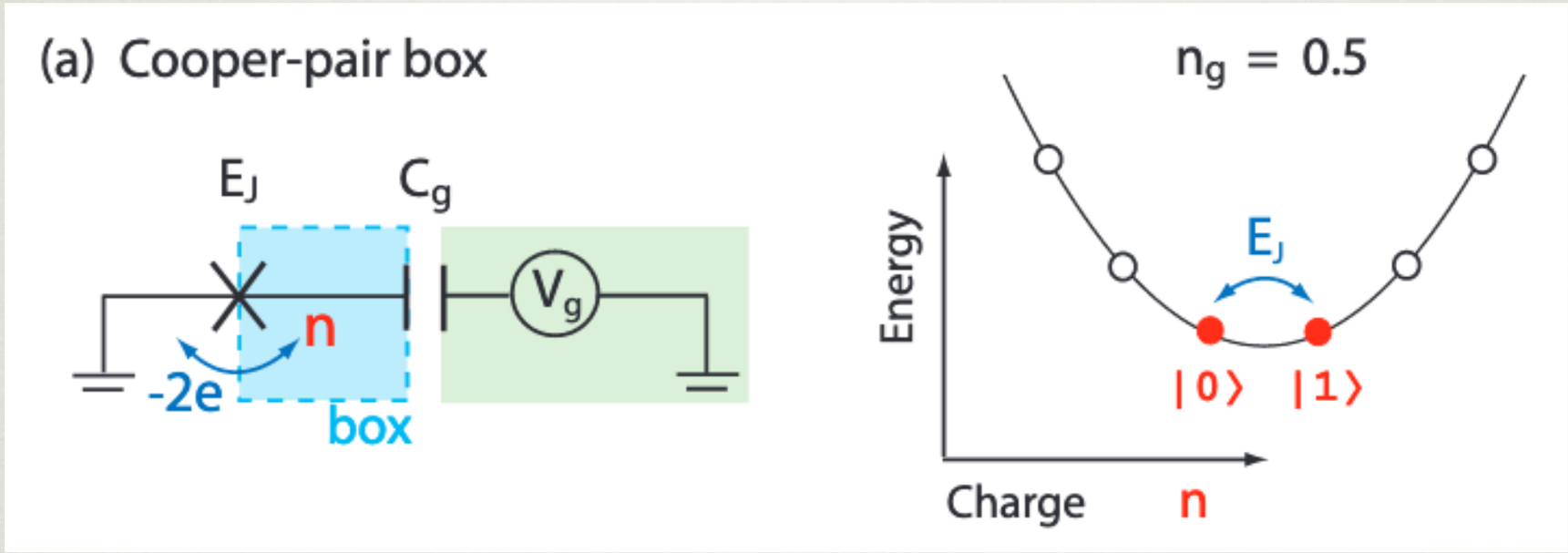
$$\mathcal{H} = 4E_C(q - q_{\text{ext}})^2 - E_J \cos\varphi + \frac{E_L}{2}(\varphi - \varphi_{\text{ext}})^2$$

$$\mathcal{H} = 4E_c\left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\varphi} - n_g\right)^2 - E_J\cos\varphi + \frac{E_L}{2}\left(\varphi - 2\pi\frac{\Phi_{\text{ext}}}{\Phi_0}\right)$$

$$\varphi_{\text{ext}} = 2\pi\frac{\Phi_{\text{ext}}}{\Phi_0}$$

Figure source: Devoret lectures

CHARGE QUBIT = COOPER PAIR BOX



$$\mathcal{H} = 4E_C(q - q_{\text{ext}})^2 - E_J \cos\varphi + \cancel{\frac{E_L}{2}(\varphi - \varphi_{\text{ext}})^2} \qquad [\varphi, q] = i$$

$$\mathcal{H} = 4E_c\left(\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}\varphi} - n_g\right)^2 - E_J\cos\varphi + \cancel{\frac{E_L}{2}\left(\varphi - 2\pi\frac{\Phi_{\text{ext}}}{\Phi_0}\right)}$$

small-coupling regime $\quad E_J \ll E_C \qquad E_L \sim 0 \qquad q_{\text{ext}} \mod 1 = \frac{1}{2}$

charge is a good quantum number

$$q = n \text{ and } q = n + 1$$

Figure source: You

$$\mathcal{H}_Q = 4E_c \sum_N (N - N_g)^2 \left|N\right\rangle \left\langle N\right| - \frac{E_J}{2} \sum_N \left(\left|N+1\right\rangle \left\langle N\right| + \text{H.c.}\right)$$

truncation to 2 lowest levels

$$\mathcal{H}_Q = -\frac{E_{\text{el}}}{2}\hat{\sigma}^x - \frac{E_J}{2}\hat{\sigma}^z \qquad E_{\text{el}} = 4E_c(1 - 2N_g)$$

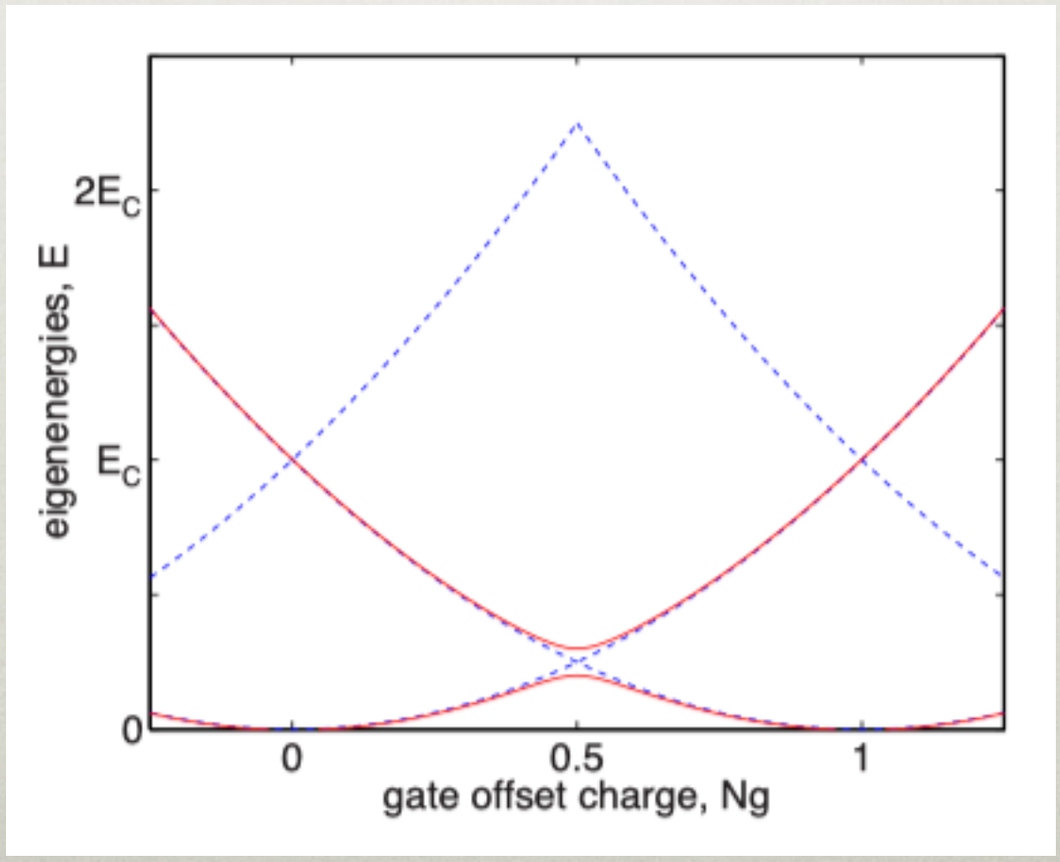


Figure source: Langform

letters to nature

(1999)

Coherent control of macroscopic quantum states in a single-Cooper-pair box

Y. Nakamura*, Yu. A. Pashkin† & J. S. Tsai*

** NEC Fundamental Research Laboratories, Tsukuba, Ibaraki 305-8051, Japan*
† CREST, Japan Science and Technology Corporation (JST), Kawaguchi, Saitama 332-0012, Japan

A nanometre-scale superconducting electrode connected to a reservoir via a Josephson junction constitutes an artificial two-level electronic system: a single-Cooper-pair box. The two levels consist of charge states (differing by 2*e*, where *e* is the electronic charge) that are coupled by tunnelling of Cooper pairs through the junction. Although the two-level system is macroscopic, containing a large number of electrons, the two charge states can be coherently superposed[1–4]. The Cooper-pair box has therefore been suggested[5–7] as a candidate for a quantum bit or 'qubit'—the basic component of a quantum computer. Here we report the observation of quantum oscillations in a single-Cooper-pair box. By applying a short voltage pulse via a gate electrode, we can control the coherent quantum state evolution: the pulse modifies the energies of the two charge states non-adiabatically, bringing them into resonance. The resulting state—a superposition of the two charge states—is detected by a tunnelling current through a probe junction. Our results demonstrate electrical coherent control of a qubit in a solid-state electronic device.

Rabi Oscillations in a Josephson-Junction Charge Two-Level System

Y. Nakamura, Yu. A. Pashkin, and J. S. Tsai
Phys. Rev. Lett. **87**, 246601 – Published 27 November 2001

Article | References | Citing Articles (155) | PDF | Export Citation

ABSTRACT

We investigated temporal behavior of an artificial two-level system driven by a strong oscillating field; namely, quantum-state evolution between two charge states in a small Josephson-junction circuit irradiated with microwaves. Rabi oscillations corresponding to 0-, 1-, and 2-photon resonances were observed. As a function of microwave amplitude, the Rabi frequencies followed a first-kind Bessel function of the corresponding order to the number of photons.

TRANSMON

$$E_J \gg E_C \qquad \sqrt{\langle \varphi^2 \rangle} = \left(\frac{2E_C}{E_J} \right)^{1/4}$$

phase is a good quantum number

condition for small phase fluctuations $E_C \ll \hbar\omega_{10}$

$$\omega_{10} = \frac{1}{\hbar}\sqrt{8E_C(E_L + E_J \cos\varphi_0)} \longrightarrow \omega_{10} = \frac{1}{\hbar}\sqrt{8E_C E_J}$$

Josephson plasma frequency

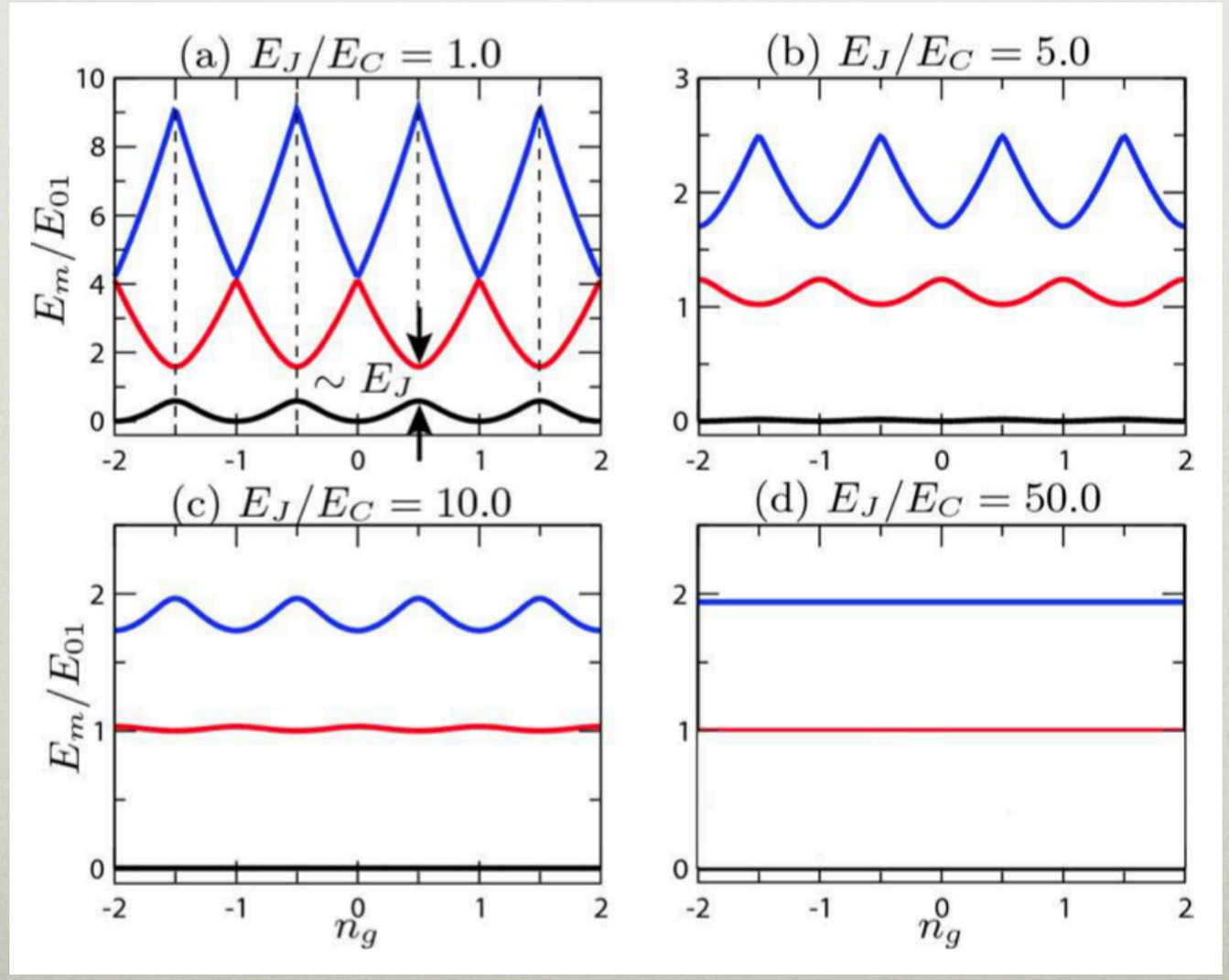


Figure source: Koch

ANHARMONICITY

$$\cos\hat{\varphi} = \text{const} - \frac{1}{2}\hat{\varphi}^2 + \frac{1}{24}\hat{\varphi}^4$$

$$\hat{\varphi} = \varphi_{\text{ZPF}}(a + a^\dagger) \qquad \varphi_{\text{ZPF}}^2 = \frac{2E_C}{E_J}$$

$$V = -\frac{E_C}{12}(a + a^\dagger)^4 \approx -\frac{E_C}{2}(a^\dagger a^\dagger a a + 2a^\dagger a)$$

weakly anharmonic oscillator $\mathcal{H} = \omega a^\dagger a + \frac{\alpha}{2} a^\dagger a^\dagger a a$

$$\hbar\omega = \sqrt{8E_J E_C} - E_C$$

$$\alpha = \omega_{21} - \omega_{10} \longrightarrow \alpha = -E_c$$

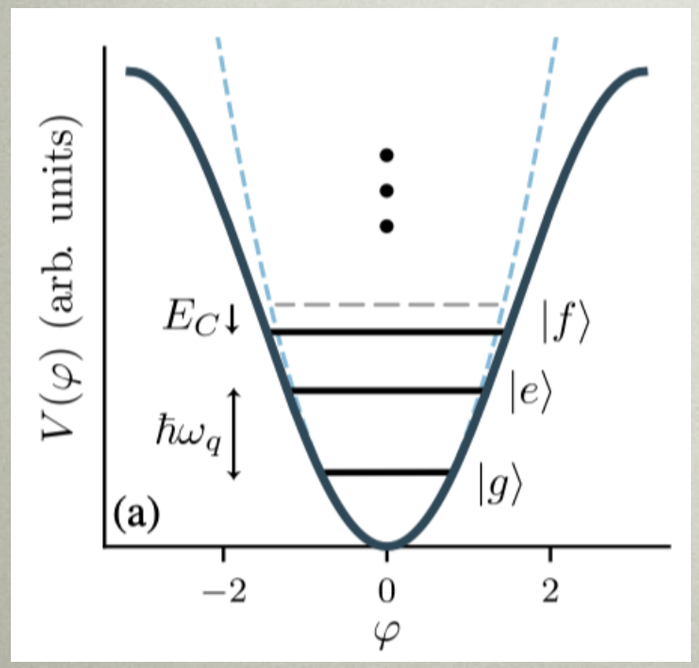


Figure source: Blais et al. RMP

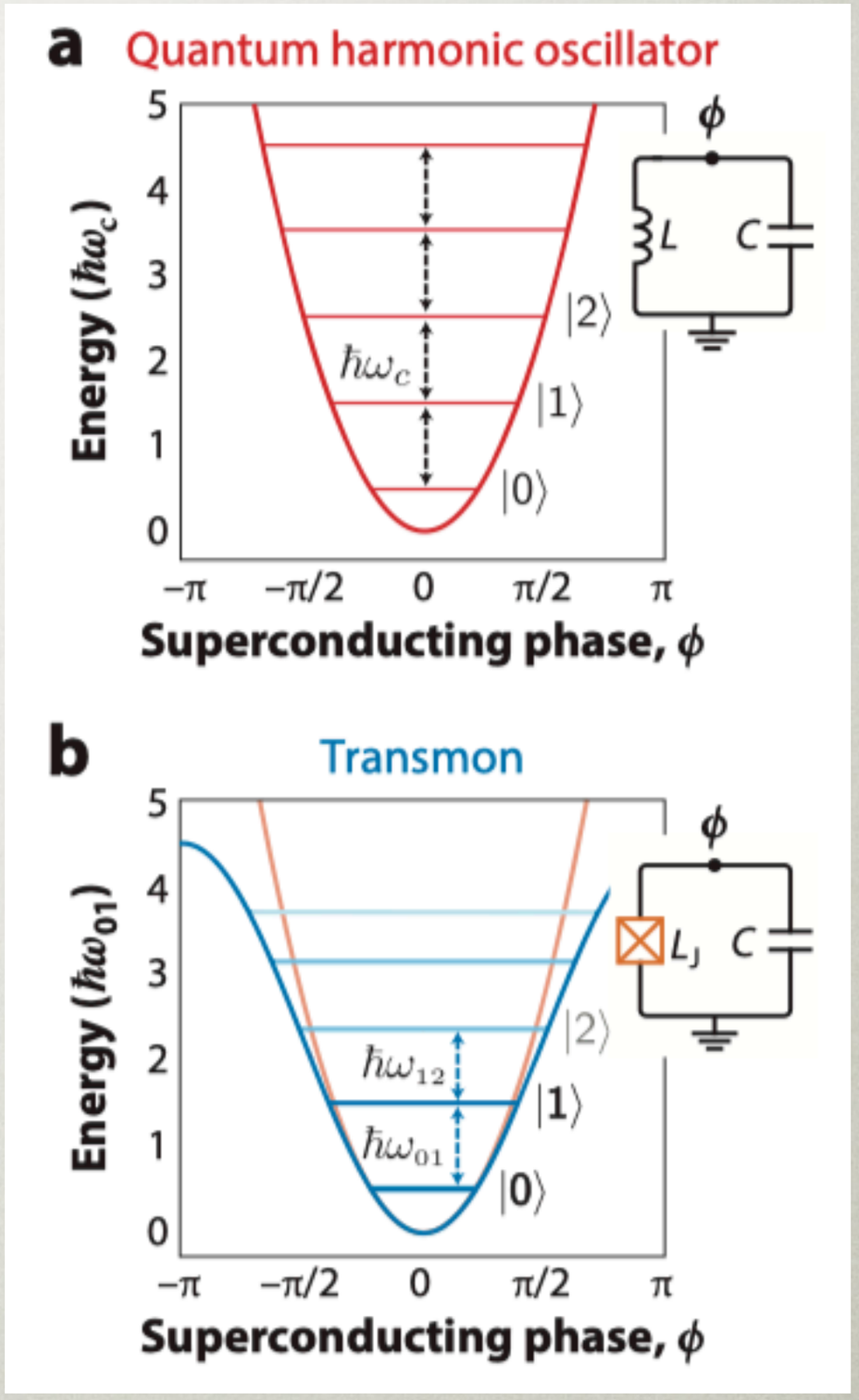


Figure source: Kjaergaard et al.

FLUX QUBIT = PERSISTENT-CURRENT QUBIT

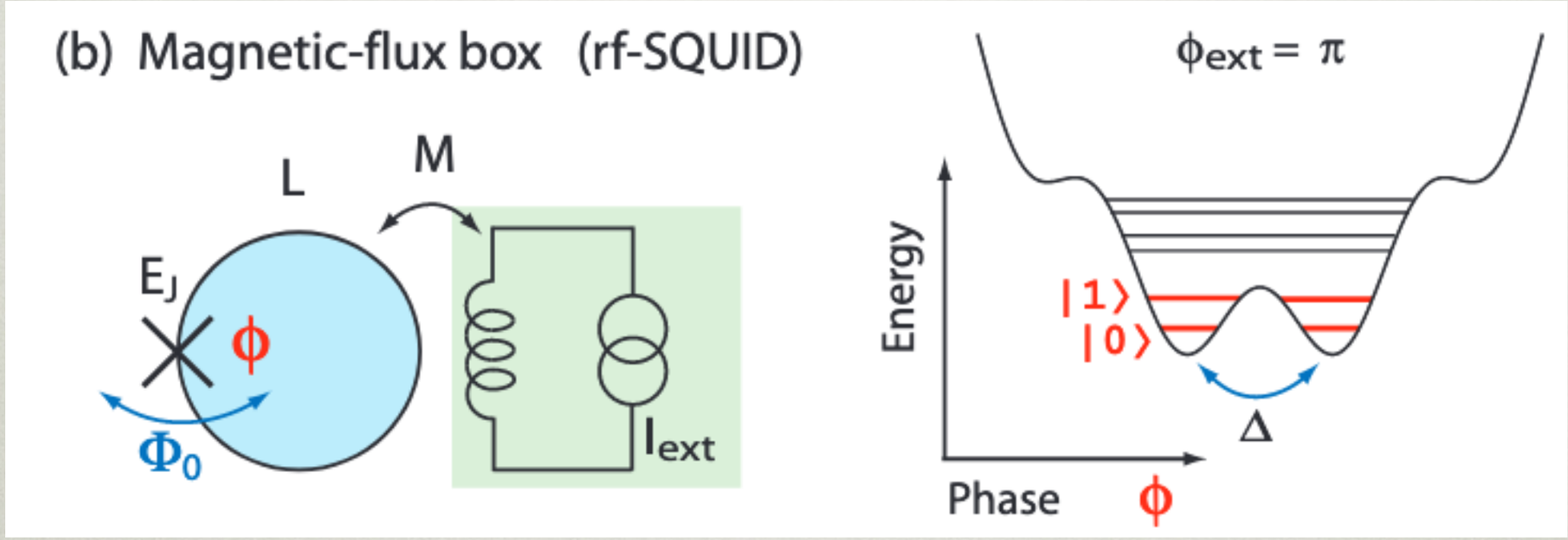
$$E_J \ll E_L$$

E_C small

$$\mathcal{H} = 4E_C N^2 - E_J \cos\varphi + \frac{1}{2} E_L (\varphi - \varphi_{\text{ext}})^2$$

$$E_J \sin\varphi_0 + E_L(\varphi_0 - \varphi_{\text{ext}}) = 0$$

$$\mathcal{H}_\varphi \approx \mathcal{H}_\varphi^{(2)} = 4E_C N^2 + \frac{1}{2}(E_L + E_J \cos\varphi_0)(\varphi - \varphi_0)^2$$



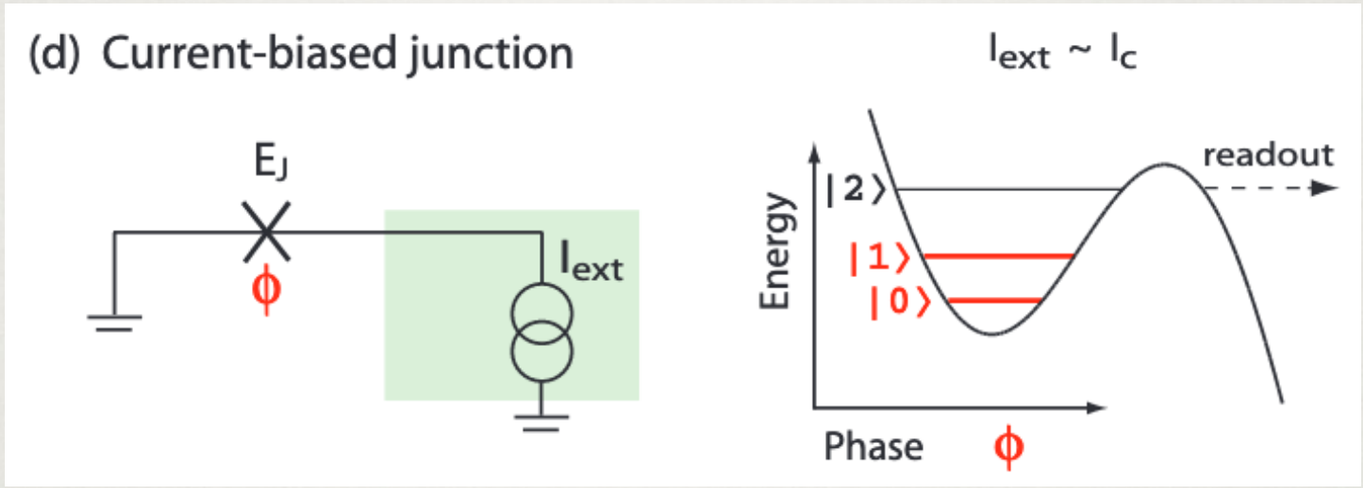
$$\Phi_{\text{ext}} \mod \Phi_0 = \Phi_0/2$$

Exact degeneracy lifted by the E_c term

Figure source: You

PHASE QUBIT = CURRENT-BIASED JOSEPHSON-JUNCTION

$$E_J \gg E_C$$



Trick: very large inductor preloaded with a very large flux, so that $\Phi/L = I$

$$\mathcal{H} = \frac{q^2}{2C} - I\phi - E_J \cos\frac{2e}{\hbar}\phi$$

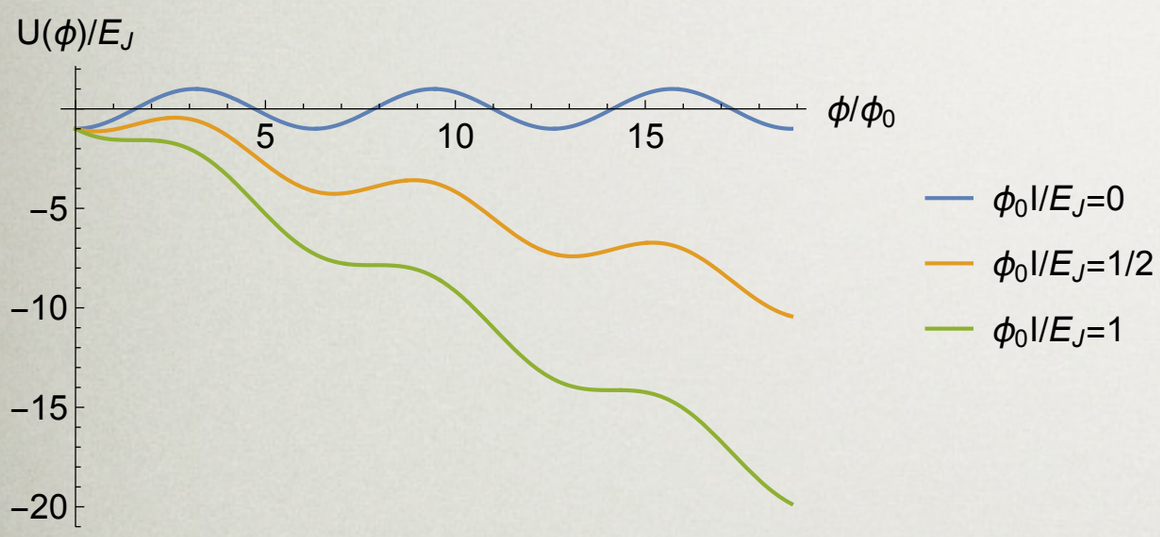
Figure source: You

WASHBOARD POTENTIAL

Current-biased Josephson junction

$$\mathcal{H} = \frac{q^2}{2C} - I\phi - E_J \cos\frac{2e}{\hbar}\phi$$

$$U(\phi) = -I\phi - E_J \cos\frac{2e}{\hbar}\phi$$



macroscopic quantum tunneling (MQT)

(1988)

Quantum Mechanics of a Macroscopic Variable: The Phase Difference of a Josephson Junction

JOHN CLARKE, ANDREW N. CLELAND, MICHEL H. DEVORET, DANIEL ESTEVE, JOHN M. MARTINIS

Experiments to investigate the quantum behavior of a macroscopic degree of freedom, namely the phase difference across a Josephson tunnel junction, are described. The experiments involve measurements of the escape rate of the junction from its zero voltage state. Low temperature measurements of the escape rate for junctions that are either nearly undamped or moderately damped agree very closely with predictions for macroscopic quantum tunneling, with no adjustable parameters. Microwave spectroscopy reveals quantized energy levels in the potential well of the junction in excellent agreement with quantum-mechanical calculations. The system can be regarded as a "macroscopic nucleus with wires."

emphasized, one must distinguish carefully between macroscopic quantum phenomena originating in the superposition of a large number of microscopic variables and those displayed by a single macroscopic degree of freedom. It is the latter that we discuss in this article.

Our usual observations on a billiard ball or Brownian particle reveal classical behavior because Planck's constant $\hbar$ is so tiny. However, at least in principle there is nothing to prevent us from designing an experiment in which these objects are quantum mechanical. To do so we have to satisfy two criteria: (i) the thermal energy must be small compared with the separation of the quantized energy levels, and (ii) the macroscopic degree of freedom must be sufficiently decoupled from all other degrees of freedom if the lifetime of the quantum states is to be longer than the characteristic time scale of the system (*1*). To illustrate the application of these

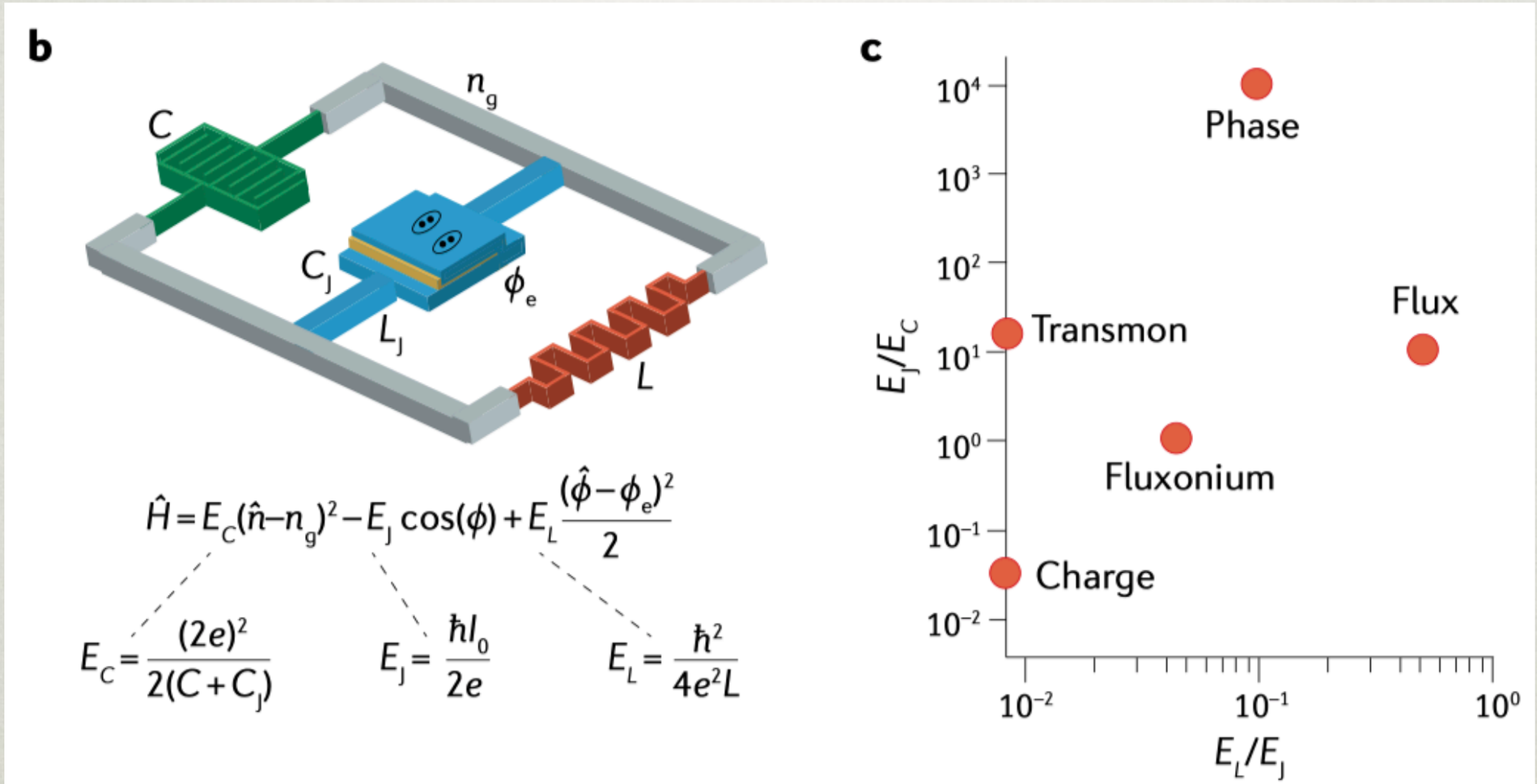


Figure source: Siddiqi

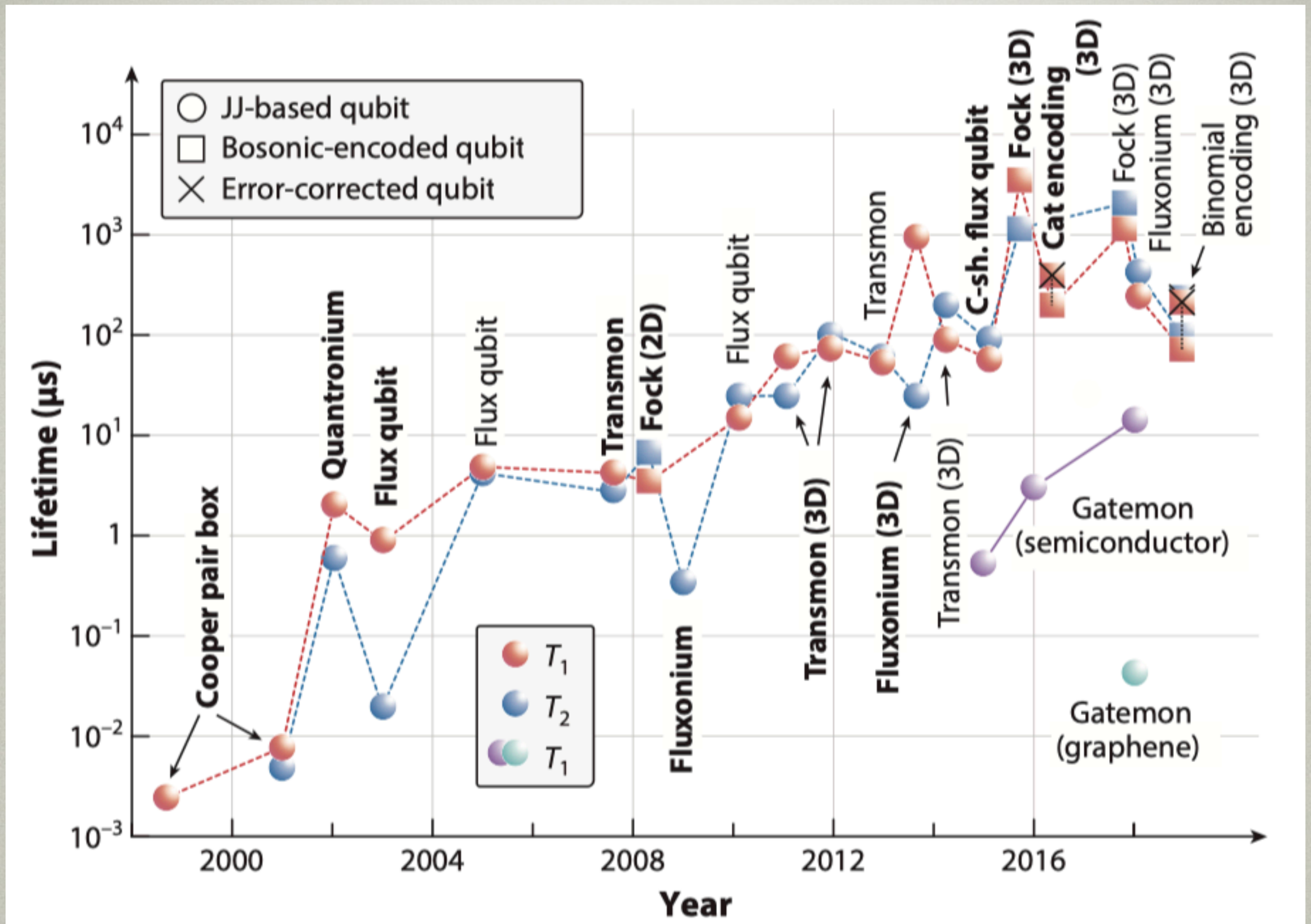


Figure source: Kjaergaard et al.

QUBIT COUPLING

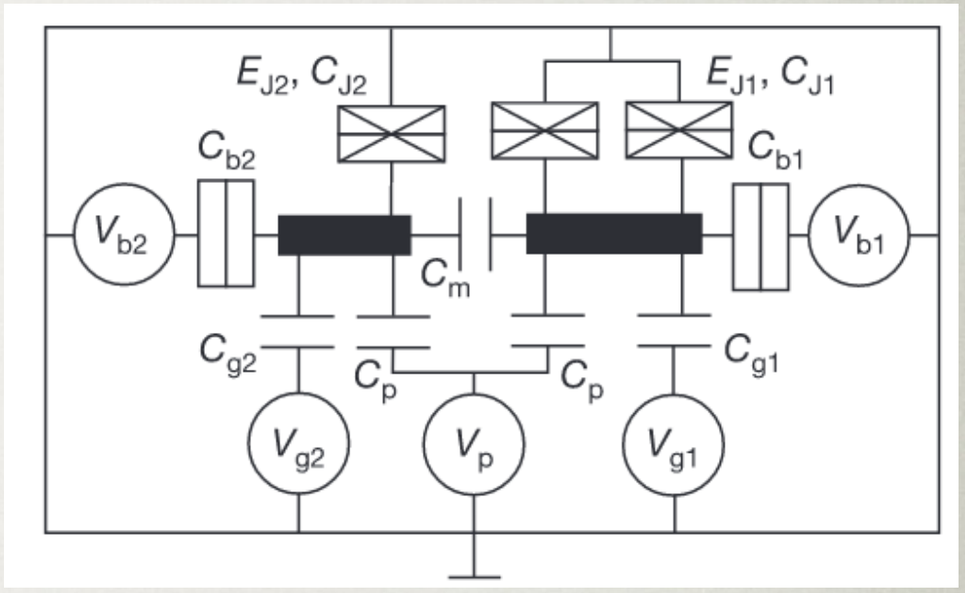
Capacitive (transverse) coupling

$$\mathcal{H}_{\text{int}} = C_g V_1 V_2$$

$$\mathcal{H} = \sum_{i=1,2} \left[4E_{C,i} n_i^2 - E_{J,i}\cos(\phi_i)\right] + 4e^2 \frac{C_g}{C_1 C_2} n_1 n_2$$

$$\mathcal{H} = \sum_{i=1,2} \left[\omega_i a_i^\dagger a_i + \frac{\alpha_i}{2} a_i^\dagger a_i^\dagger a_i a_i\right] - g(a_1 - a_1^\dagger)(a_2 - a_2^\dagger)$$

$$\mathcal{H} = \sum_{i=1,2} \frac{\omega_i}{2}\sigma_{z,i} + g\sigma_{y,1}\sigma_{y,2}$$



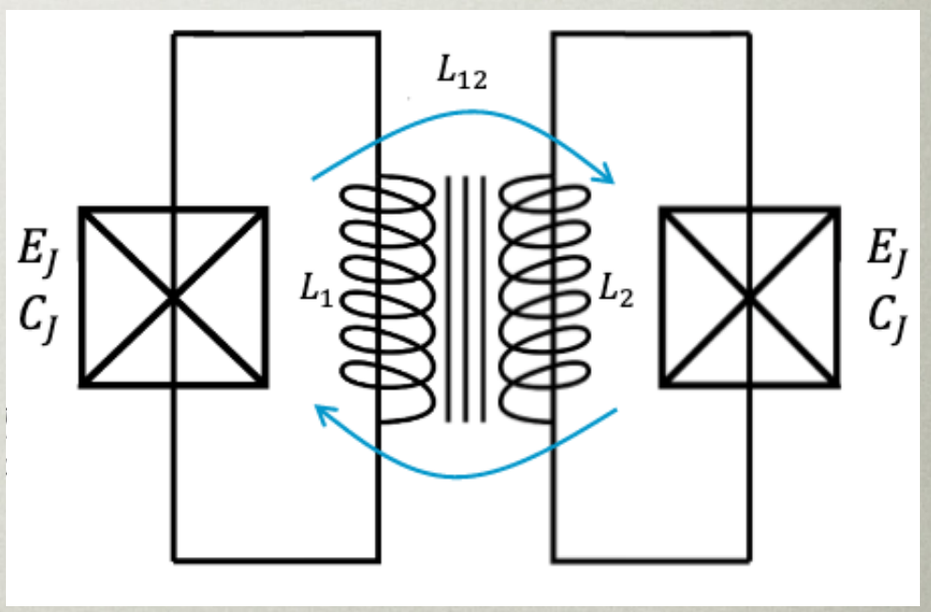
Pashkin et al. (2003)

Inductive (longitudinal) coupling

$$\mathcal{H}_{\text{int}} = M_{12} I_1 I_2$$

$$\mathcal{H} = \sum_{i=1,2} \left[4E_{C,i} n_i^2 + \frac{E_{L,i}}{2}\phi_i^2 - E_{J,i}\cos(\phi_i)\right] + M_{12} I_{c1}\sin(\phi_1) I_{c2}\sin(\phi_2)$$

$$\mathcal{H} = \sum_{i=1,2} \frac{\omega_i}{2}\sigma_{z,i} + g\sigma_{z,1}\sigma_{z,2}$$



CIRCUIT QUANTUM ELECTRODYNAMICS (cQED)

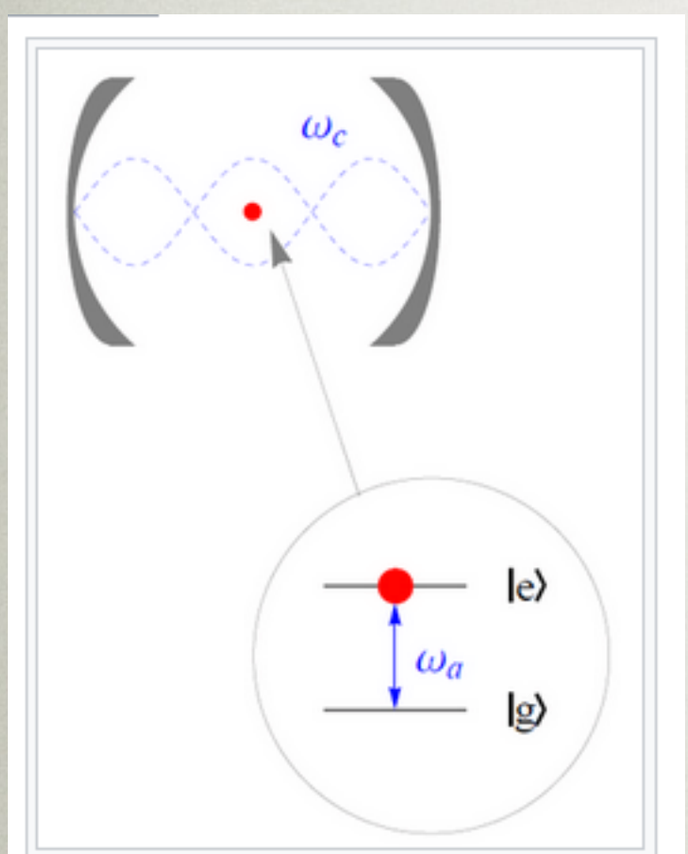
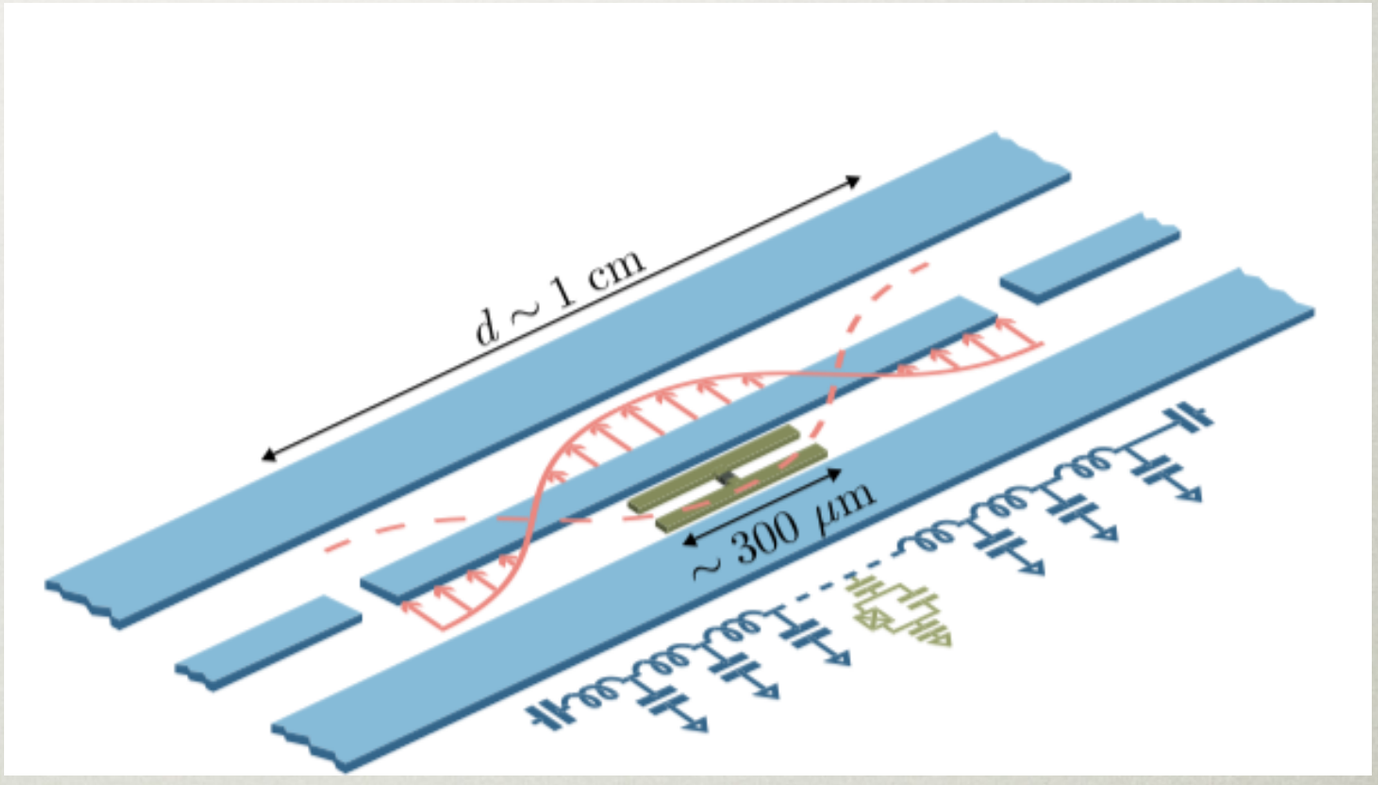


Illustration of the Jaynes-Cummings model. An atom in an optical cavity is shown as red dot on the top left. The energy levels of the atom that couple to the field mode within the cavity are shown in the circle on the bottom right. Transfer between the two states causes photon emission (absorption) by the atom into (out of) the cavity mode.

cavity QED



circuit QED

Figure source: Blais et al. RMP

SUPERCONDUCTING TRANSMISSION LINES

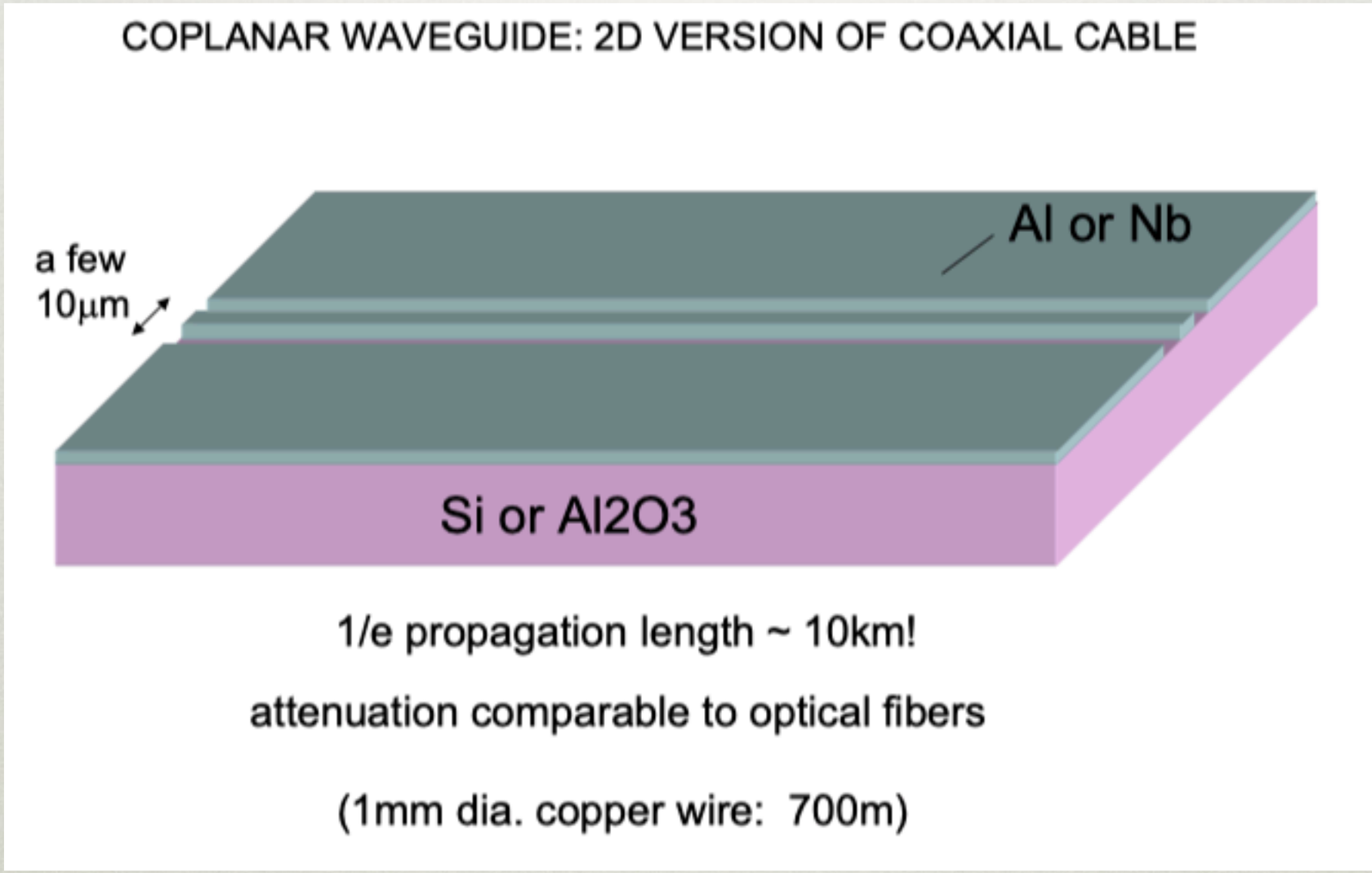


Figure source: Devoret lectures

COPLANAR WAVEGUIDE RESONATOR

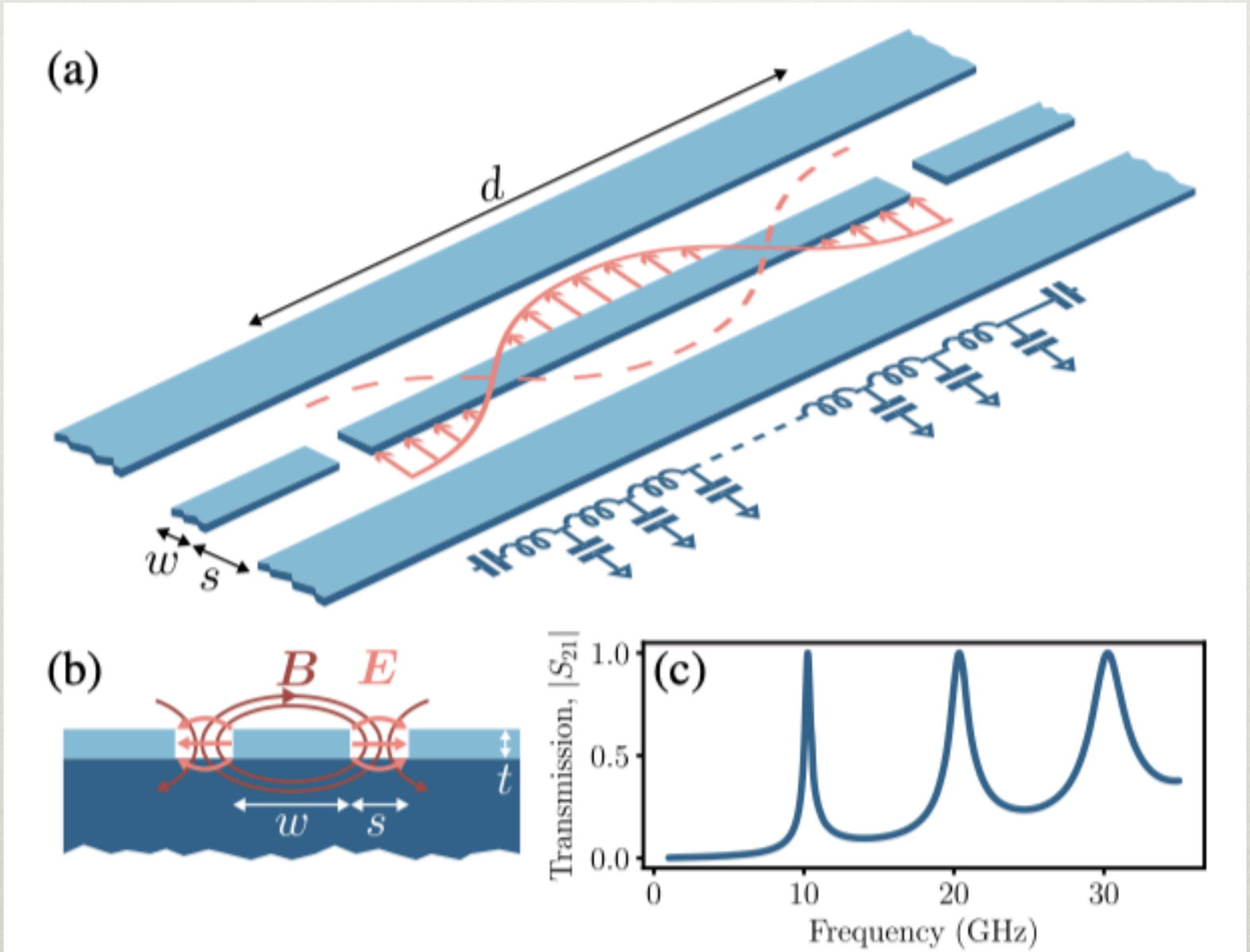


Figure source: Blais et al. RMP

VECTOR NETWORK ANALYZER



VECTOR NETWORK ANALYZER

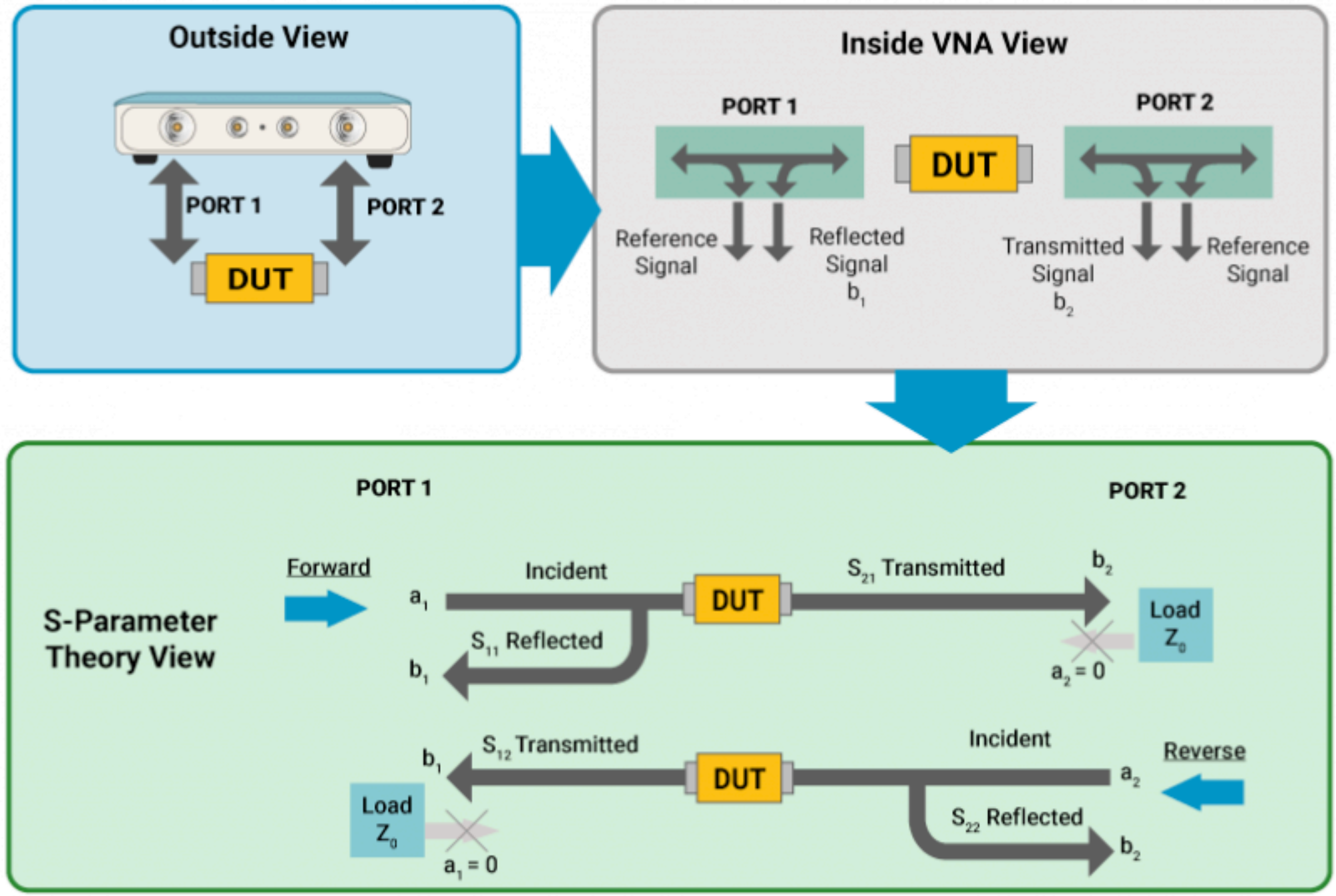
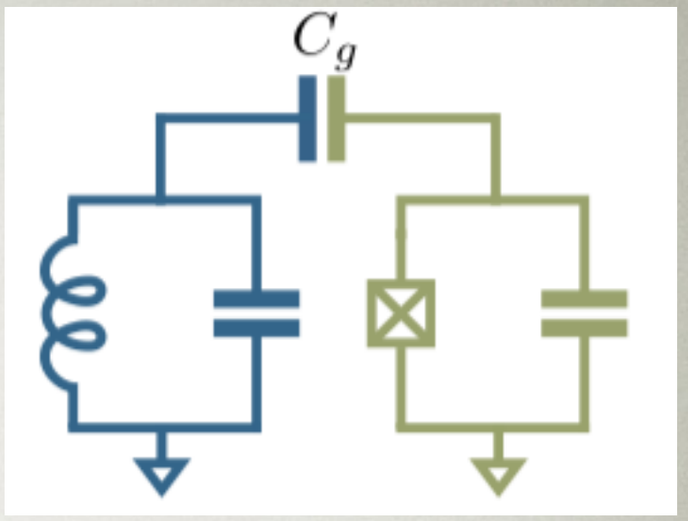


Figure source: Sierra circuits

QUBIT-RESONATOR COUPLING



Jaynes-Cummings Hamiltonian

$$\mathcal{H}_{\rm JC} = \hbar\omega_r \left(a^\dagger a + \frac{1}{2}\right) + \hbar\Omega\frac{\sigma_z}{2} + \hbar g(a^\dagger\sigma^- + \sigma^+ a) + \mathcal{H}_\kappa + \mathcal{H}_\gamma$$

$$|\overline{+,n}\rangle = \cos\theta_n\,|\uparrow,n\rangle + \sin\theta_n\,|\downarrow,n+1\rangle$$
$$|\overline{-,n}\rangle = -\sin\theta_n\,|\uparrow,n\rangle + \cos\theta_n\,|\downarrow,n+1\rangle$$

$$\theta_n = \frac{1}{2}\tan^{-1}\left(\frac{2g\sqrt{n+1}}{\Delta}\right)$$

$$E_{\overline{\pm},n} = (n+1)\hbar\omega_r \pm \frac{\hbar}{2}\sqrt{4g^2(n+1)+\Delta^2}$$

$$E_{\downarrow,0} = -\frac{\hbar\Delta}{2}$$

$\Delta \equiv \Omega - \omega_r$ detuning

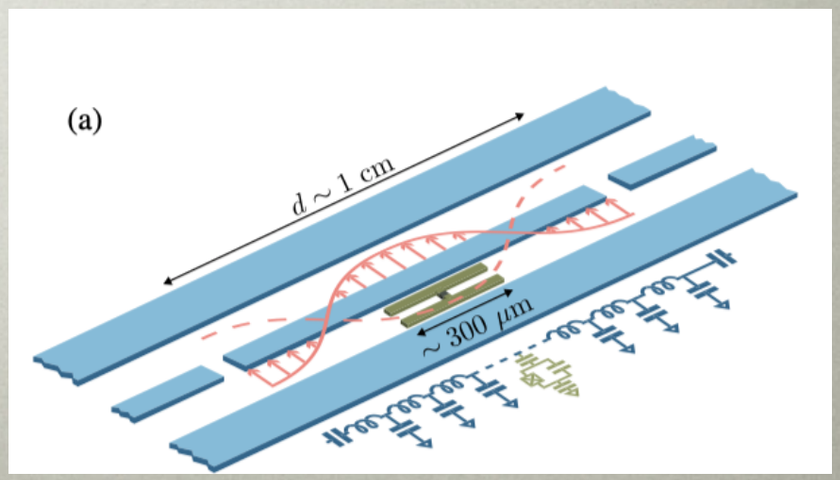


Figure source: Blais et al. RMP

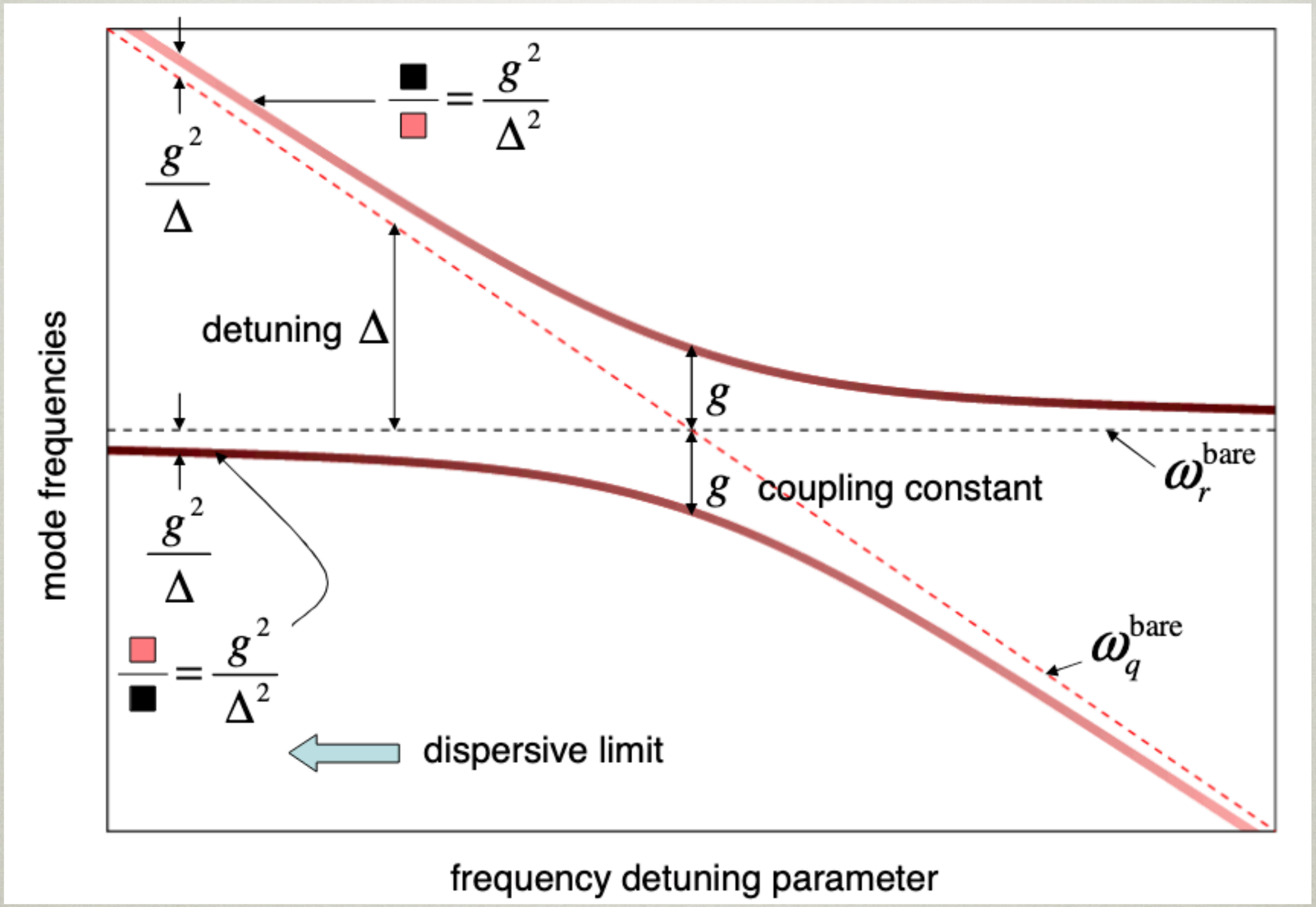
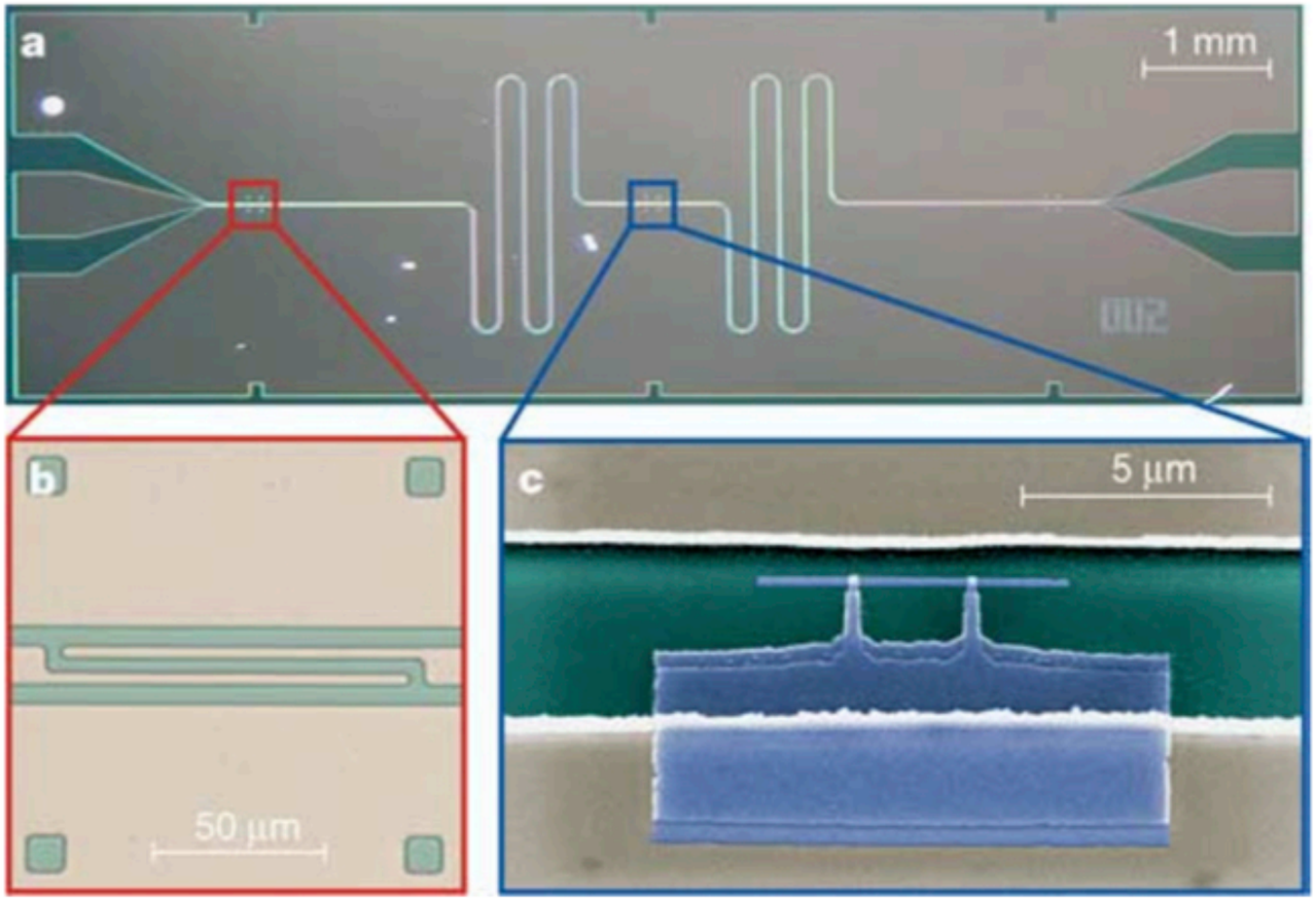


Figure source: Devoret lectures

STRONG COUPLING EXPERIMENT

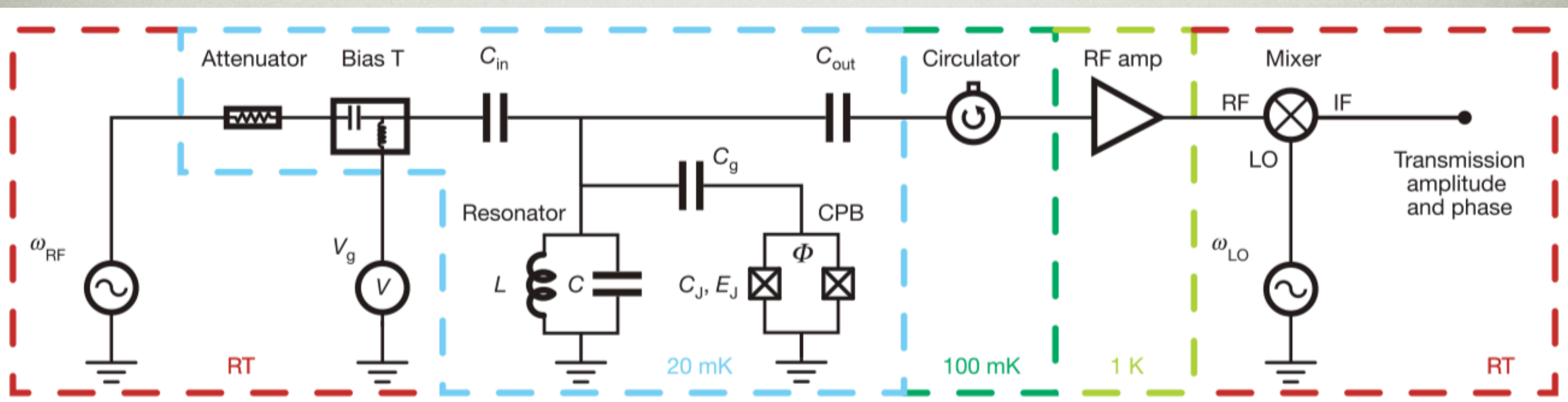


Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics

A. Wallraff[1], D. I. Schuster[1], A. Blais[1], L. Frunzio[1], R.- S. Huang[1,2], J. Majer[1], S. Kumar[1], S. M. Girvin[1] & R. J. Schoelkopf[1]

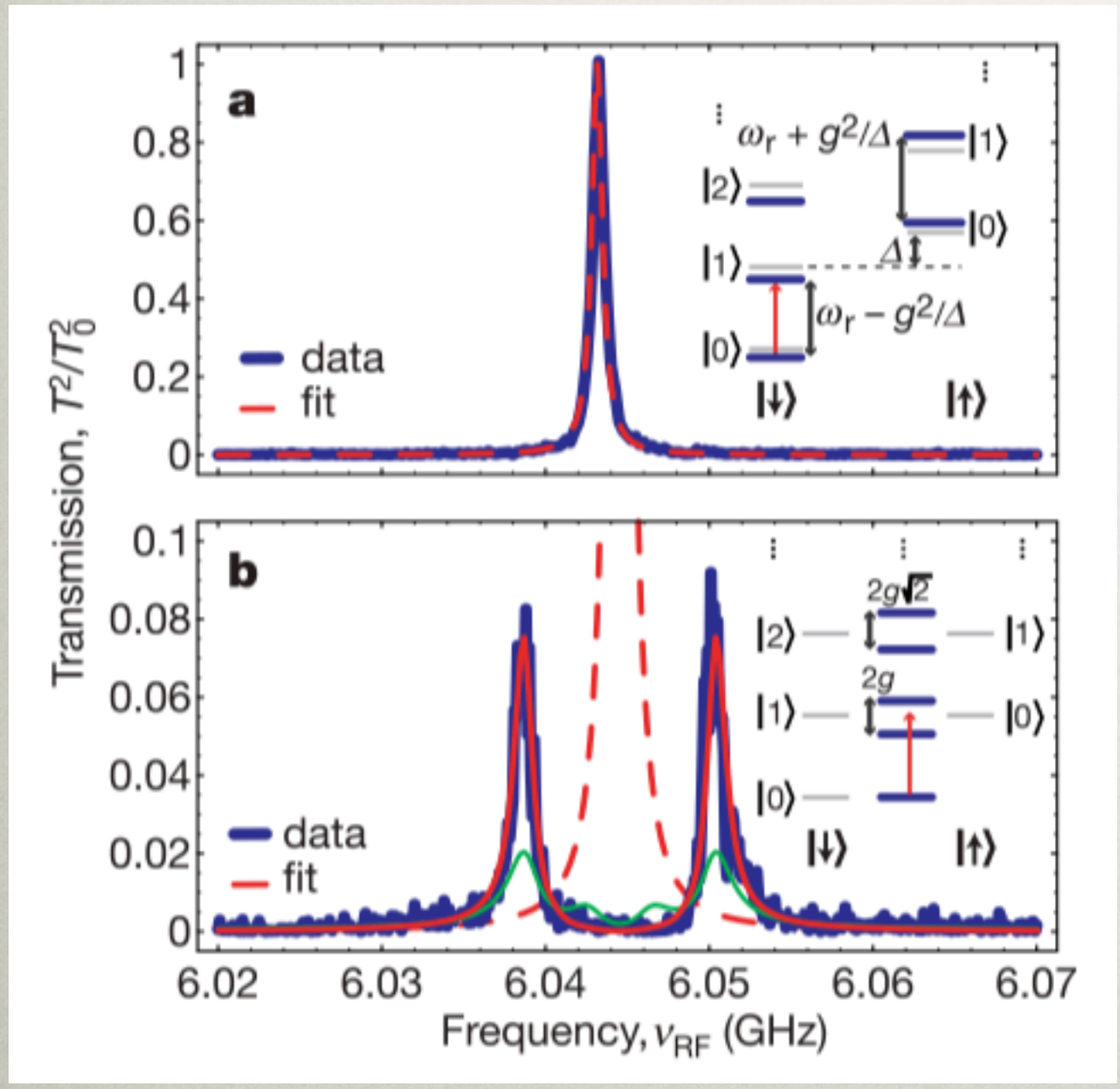
[1]*Departments of Applied Physics and Physics, Yale University, New Haven, Connecticut 06520, USA*
[2]*Department of Physics, Indiana University, Bloomington, Indiana 47405, USA*

(2004)



VACUUM RABI SPLITTING

"Atom" alternatively emits photons and reabsorbs them.
Spontaneous emission is a consequence of coupling between the atom and the vacuum fluctuations of the field.



in resonance, $\Delta = 0$

$$|\overline{\pm,0}\rangle = \frac{|\uparrow,1\rangle \pm |\downarrow,0\rangle}{\sqrt{2}}$$

$$E_{\mathrm{Rabi}} = \frac{\sqrt{n}g}{\pi}$$

DISPERSIVE CASE

large detuning, $\Delta \gg g$

$$\left|\overline{-,0}\right\rangle \sim -(g/\Delta)\left|\uparrow,0\right\rangle + \left|\downarrow,1\right\rangle$$
$$\left|\overline{+,0}\right\rangle \sim \left|\uparrow,0\right\rangle + (g/\Delta)\left|\downarrow,1\right\rangle$$

$$U = \exp\left[\frac{g}{\Delta}(a\sigma^{+} - a^{\dagger}\sigma^{-})\right]$$

$$U\mathcal{H}U^{\dagger} \approx \hbar\left[\omega_r + \frac{g^2}{\Delta}\sigma^z\right]a^{\dagger}a + \hbar\left[\Omega + \frac{g^2}{\Delta}\right]\frac{\sigma^z}{2}$$

dispersive shift $\quad \chi = \dfrac{g^2}{\Delta}$

Lamb shift $\quad \tilde{\Omega} = \Omega + \dfrac{g^2}{\Delta}$

$$\mathcal{H} = \omega_r\left(a^{\dagger}a + \frac{1}{2}\right) + \frac{1}{2}\left(\Omega + \frac{g^2}{\Delta} + \frac{2g^2}{\Delta}a^{\dagger}a\right)\sigma^z$$

ac-Stark shift

[Submitted on 1 Feb 2022]

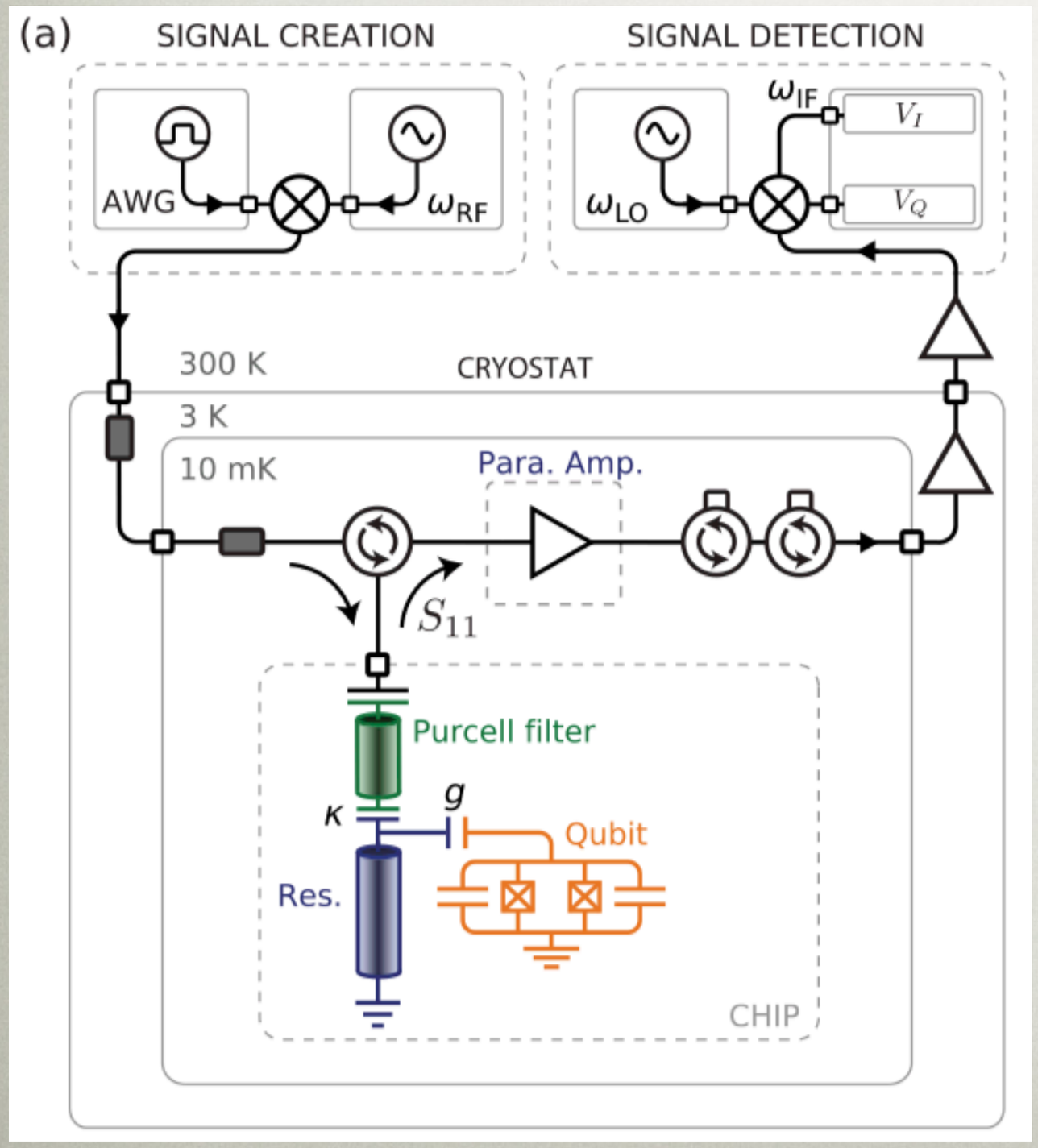
The Jaynes-Cummings model and its descendants

Jonas Larson, Th. K. Mavrogordatos

The Jaynes-Cummings (JC) model has been at the forefront of quantum optics for almost six decades to date, providing one of the simplest yet intricately nonlinear formulations of light-matter interaction in modern physics. Laying most of the emphasis to the omnipresence of the model across a range of disciplines, this monograph brings up the fundamental generality of its formalism, looking at a wide gamut of applications in specific physical systems among several realms, including atomic physics, quantum optics, solid-state physics and quantum information science. When bringing the various pieces together to assemble our narrative, we have primarily targeted researchers in quantum physics and quantum optics. The monograph also comprises an accessible introduction for graduate students engaged with non-equilibrium quantum phase transitions, quantum computing and simulation, and quantum many-body physics. In that framework, we aim to reveal the common ground between physics and applications scattered across literature and different technological advancements. The exposition guides the reader through a vibrant field interlacing quantum optics and condensed-matter physics. All sections are devoted to the strong interconnection between theory and experiment, historically linked to the development of the various modern research directions stemming from JC physics. This is accompanied by a comprehensive list of references to the key publications that have shaped its evolution since the early 1960s. Finally, we have endeavoured to keep the presentation of such a multi-sided material as concise as possible, interspersing continuous text with various illustrations alongside an economical use of mathematical expressions.

Comments: Published as part of IOP ebooks; ISBN: 978-0-7503-3447-1. 237 pages, 73 figures, almost 1600 references
Subjects: **Quantum Physics (quant-ph)**; Quantum Gases (cond-mat.quant-gas); Atomic Physics (physics.atom-ph)
Journal reference: IOP ebooks, 2021
DOI: 10.1088/978-0-7503-3447-1
Cite as: arXiv:2202.00330 **[quant-ph]**
(or arXiv:2202.00330v1 **[quant-ph]** for this version)

Read-out



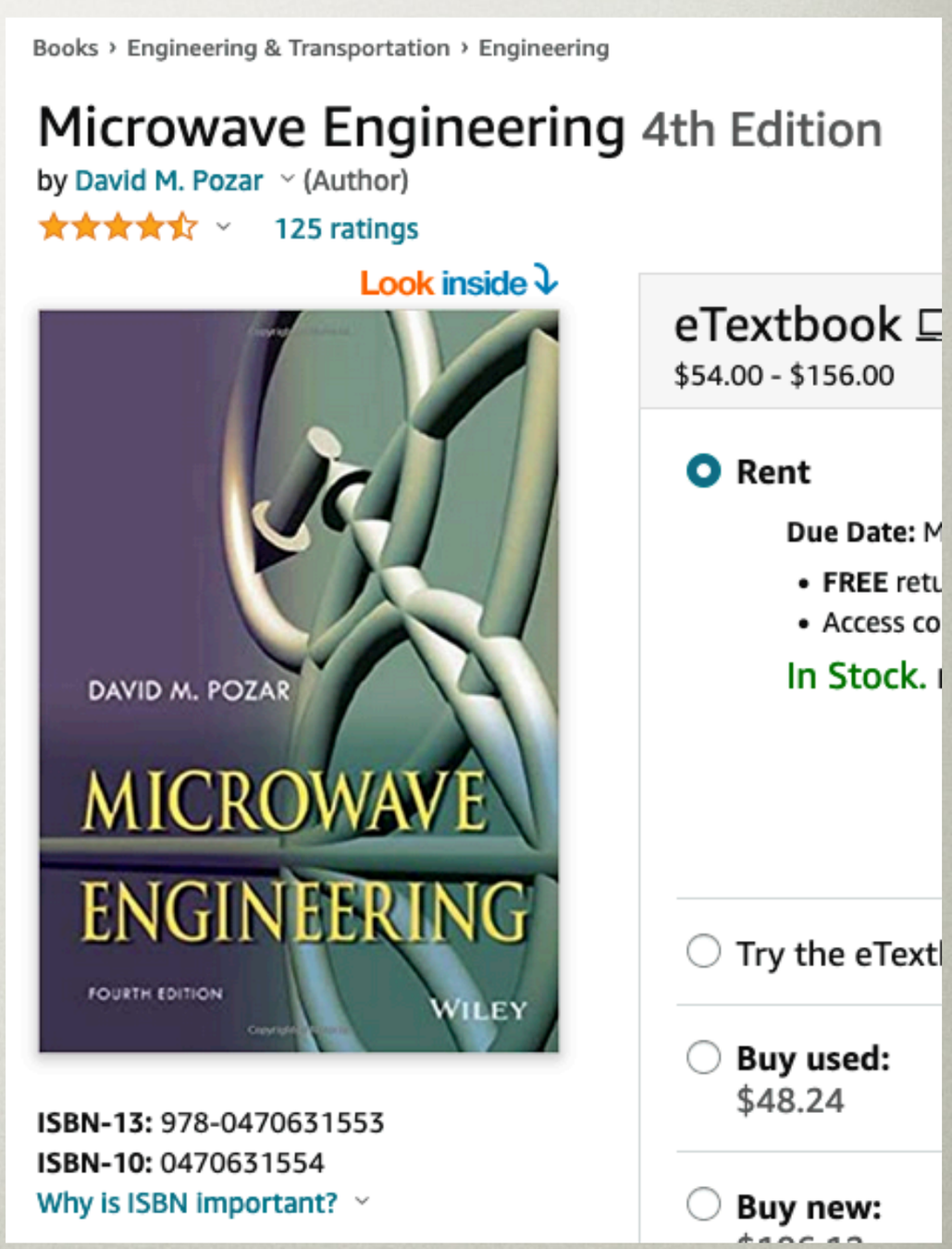
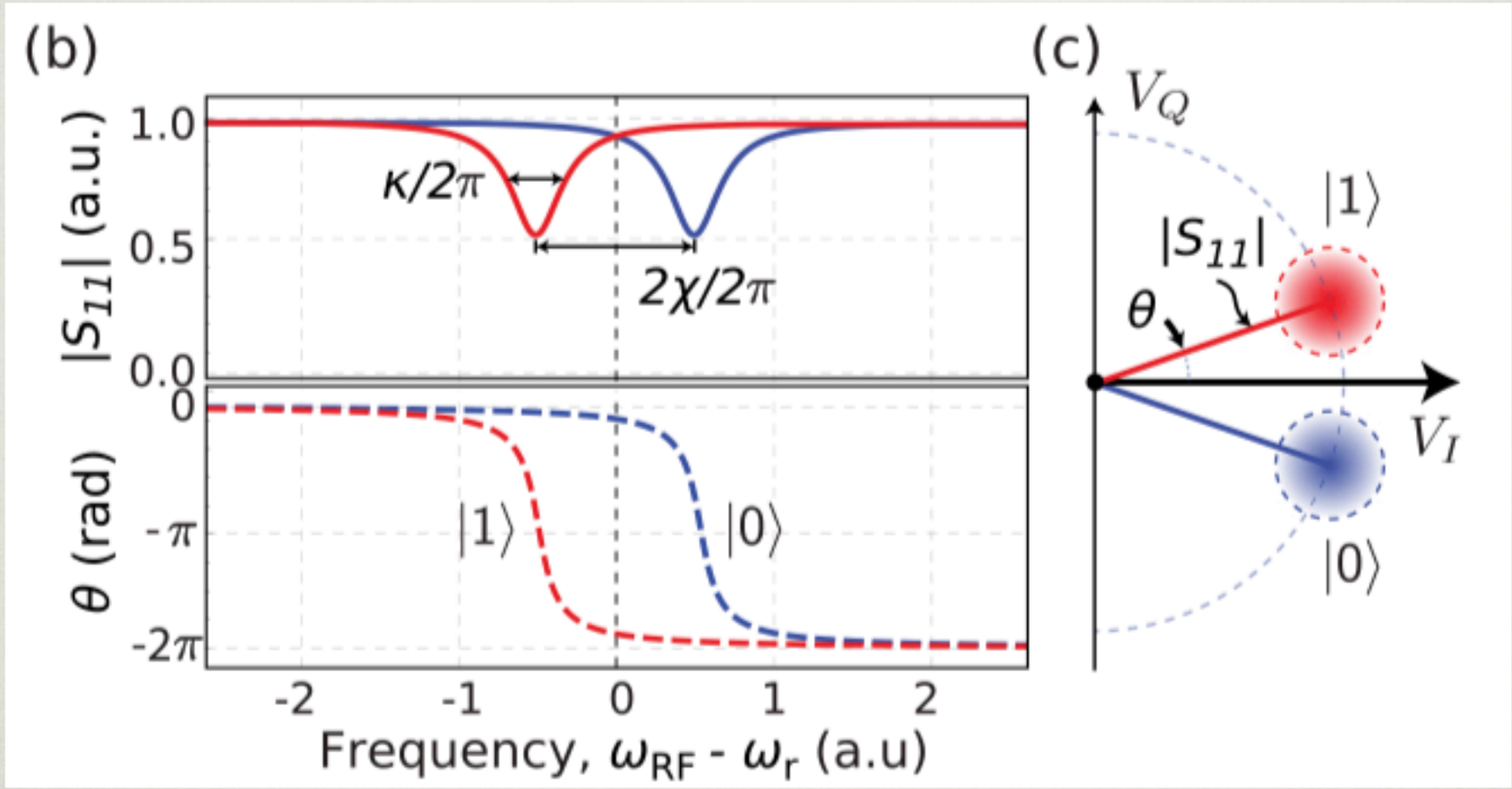


Figure source: Krantz (2021)

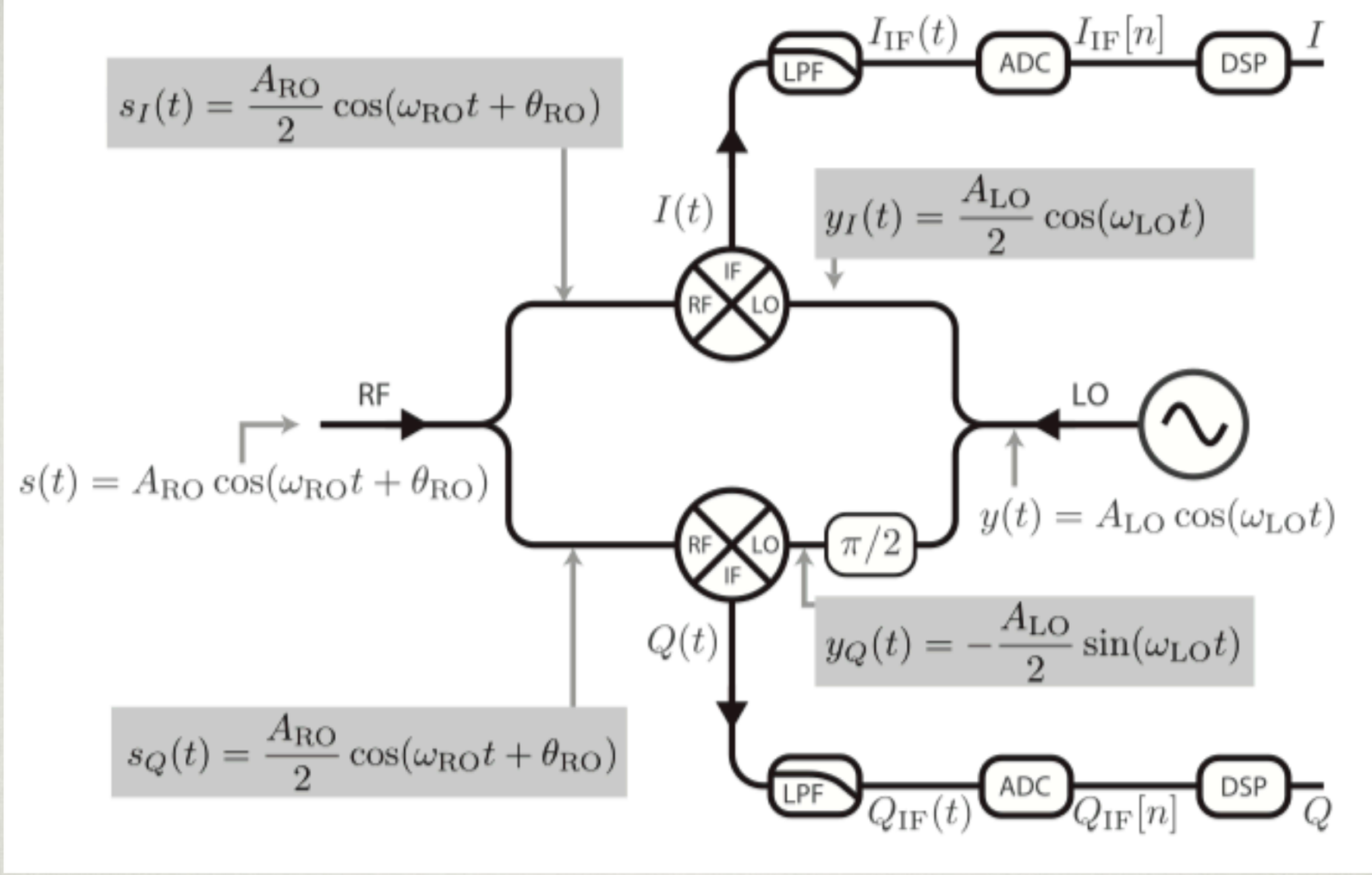
DISPERSIVE SENSING



$$\omega_{\mathrm{RF}} = \left(\omega_r^{|0\rangle} + \omega_r^{|1\rangle}\right)/2$$

I: in-phase

Q: quadrature ($\pi/2$)



read-out signal

$$
\begin{aligned}
s(t) &= A_{\mathrm{RO}}\cos(\omega_{\mathrm{RO}}t + \theta_{\mathrm{RO}}) \\
&= \mathrm{Re}\left[A_{\mathrm{RO}}e^{i\theta_{\mathrm{RO}}}e^{i\omega_{\mathrm{RO}}t}\right] \\
&= \mathrm{Re}\left[(I + iQ)e^{i\omega_{\mathrm{RO}}t}\right]
\end{aligned}
$$

local-oscillator signal

$$y(t) = A_{\mathrm{LO}}\cos(\omega_{\mathrm{LO}}t)$$

intermediate frequency

$$\omega_{\mathrm{IF}} = \omega_{\mathrm{RO}} \pm \omega_{\mathrm{LO}}$$

homodyne detection (= lock-in detector)

$$\omega_{\mathrm{LO}} = \omega_{\mathrm{RO}}$$

$$I = \frac{1}{T}\int_0^T \mathrm{d}t\, s_I(t) y_I(t) = \frac{A_{\mathrm{RO}} A_{\mathrm{LO}}}{8}\cos\theta_{\mathrm{RO}}$$

$$Q = \frac{1}{T}\int_0^T \mathrm{d}t\, s_Q(t) y_Q(t) = \frac{A_{\mathrm{RO}} A_{\mathrm{LO}}}{8}\sin\theta_{\mathrm{RO}}$$

$$A_{\mathrm{RO}} \propto \sqrt{I^2 + Q^2}$$

$$\theta_{\mathrm{RO}} = \arctan\frac{Q}{I}$$

heterodyne detection

$$I_{\mathrm{IF}}(t) = \frac{1}{T}\int_0^T \mathrm{d}t\, s_I(t) y_I(t) = \frac{A_{\mathrm{RO}} A_{\mathrm{LO}}}{8}\cos(\omega_{\mathrm{IF}} t + \theta_{\mathrm{RO}})$$

$$Q_{\mathrm{IF}}(t) = \frac{1}{T}\int_0^T \mathrm{d}t\, s_Q(t) y_Q(t) = \frac{A_{\mathrm{RO}} A_{\mathrm{LO}}}{8}\sin(\omega_{\mathrm{IF}} t + \theta_{\mathrm{RO}})$$

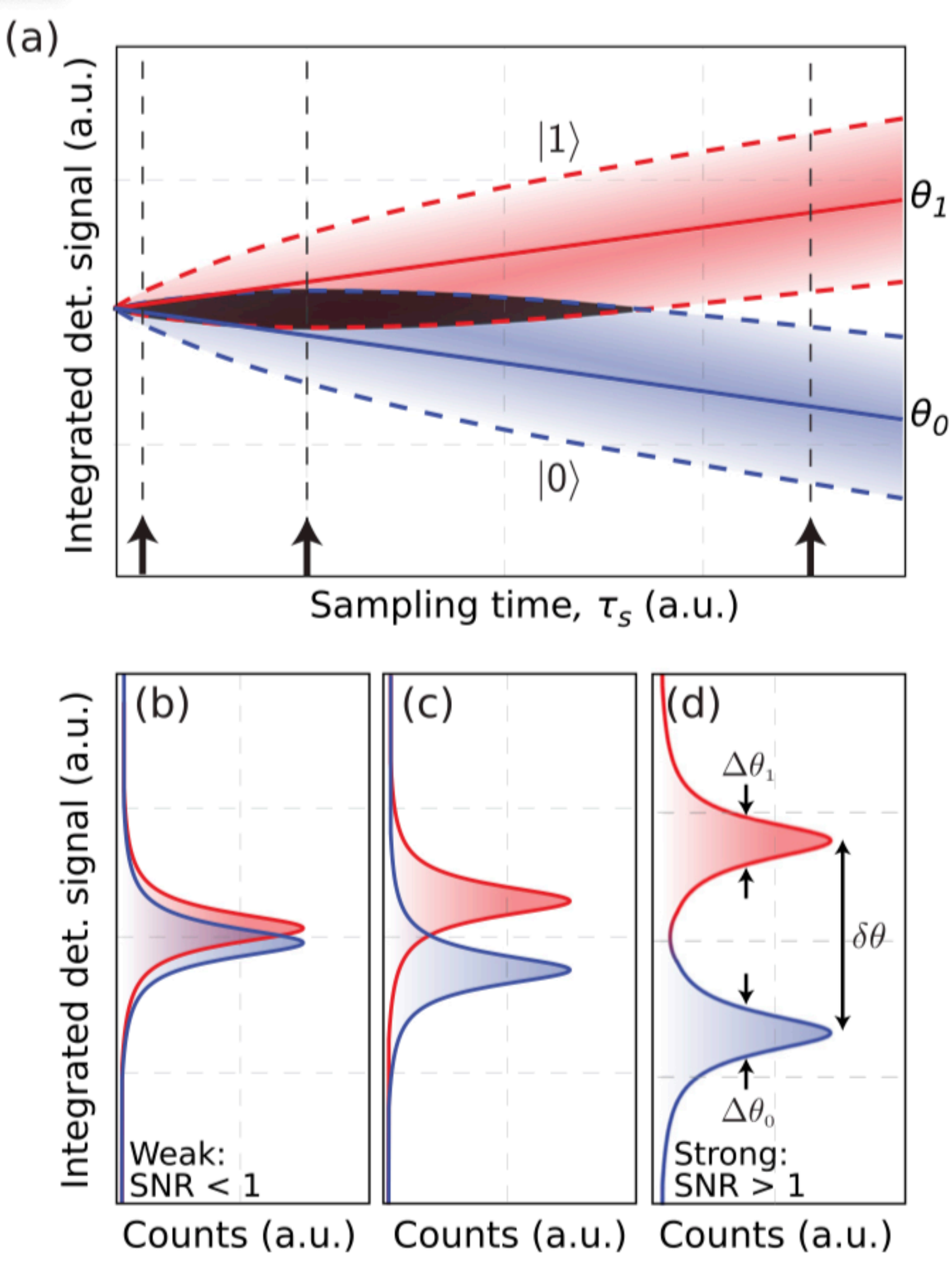
$$n = t/\Delta \quad \Omega_{\mathrm{IF}} = \omega_{\mathrm{IF}}\Delta t$$

$$I = \frac{1}{M}\sum_{n_1}^{n_2} I_{\mathrm{IF}}[n]\cos[\Omega_{\mathrm{IF}} n] = \frac{A_{\mathrm{RO}} A_{\mathrm{LO}}}{16}\cos\theta_{\mathrm{RO}}$$

$$Q = \frac{1}{M}\sum_{n_1}^{n_2} Q_{\mathrm{IF}}[n]\sin[\Omega_{\mathrm{IF}} n] = \frac{A_{\mathrm{RO}} A_{\mathrm{LO}}}{16}\sin\theta_{\mathrm{RO}}$$

$$M = n_2 - n_1 + 1$$

$$\bar{z} = \frac{A_{\mathrm{RO}} A_{\mathrm{LO}}}{8} e^{i\theta_{\mathrm{RO}}}$$

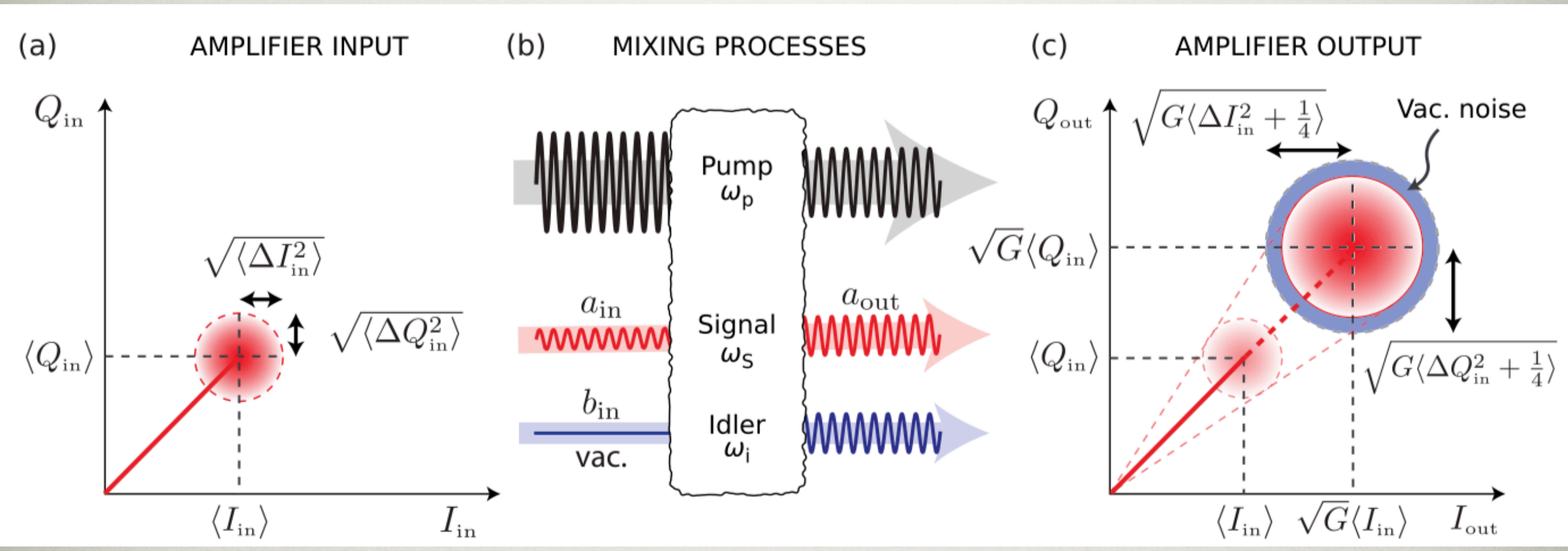


fidelity decay

$$F(\tau_{ro}) = 1 - e^{-\tau_{ro}/T_1}$$

$$\tau_{ro} = \tau_{rd} + \tau_s/2$$

AMPLIFICATION (AT QUANTUM LEVEL)



$$[a_{\rm in}, a_{\rm in}^\dagger] = 1 \quad [a_{\rm out}, a_{\rm out}^\dagger] = 1$$

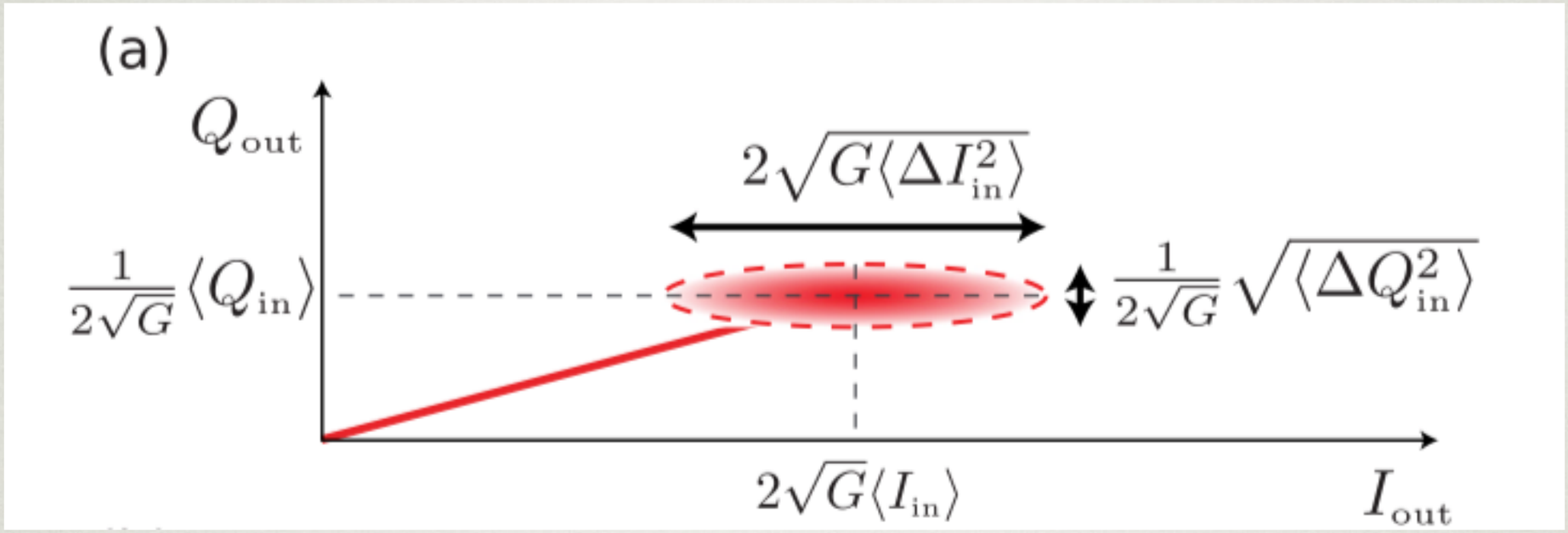
$$a_{\rm out} = \sqrt{G} a_{\rm in}$$

$$a_{\rm out} = \sqrt{G} a_{\rm in} + \sqrt{G-1} b_{\rm in}^\dagger$$

$$[b_{\rm in}, b_{\rm in}^\dagger] = 1$$

G: intensity amplification coefficient

quantum-limited amplification



$$a_{\rm out} = \sqrt{G}a_{\rm in} + e^{-\phi}\sqrt{G-1}a_{\rm in}^{\dagger}$$ single-mode squeezing

Modelling dissipation

$$\mathcal{H}_{\mathrm{JC}} = \hbar\omega_r \left(a^\dagger a + \frac{1}{2}\right) + \hbar\Omega\frac{\sigma_z}{2} + \hbar g(a^\dagger\sigma^- + \sigma^+ a) + \mathcal{H}_\kappa + \mathcal{H}_\gamma$$

Liouville-von Neumann equation for system+enviroment

$$\dot{\rho}_{\mathrm{tot}} = -\frac{i}{\hbar}\left[\mathcal{H}_{\mathrm{sys}} + \mathcal{H}_\kappa + \mathcal{H}_\gamma, \rho_{\mathrm{tot}}\right]$$

Weak coupling, EOM for system = Lindblad equation

$$\dot{\rho} = -\frac{i}{\hbar}[\mathcal{H}_{\mathrm{sys}}, \rho] - \frac{1}{2}\sum_{m=\{\kappa,\gamma\}}\left(L_m^\dagger L_m\rho + \rho L_m^\dagger L_m - 2L_m\rho L_m\right)$$

$L_\kappa = \sqrt{\kappa}a$

$L_\gamma = \sqrt{\gamma}\sigma^-$

L_m: Lindblad (jump) operators

More on this: Zala Lenarčič, Apr 6 and Apr 13
Open quantum systems

Hybrid superconductor-semiconductor devices

nature nanotechnology

REVIEW ARTICLE

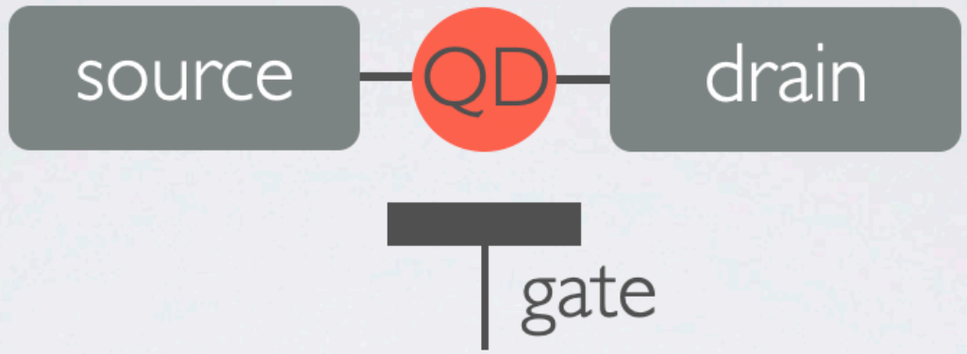
PUBLISHED ONLINE: 19 SEPTEMBER 2010 | DOI: 10.1038/NNANO.2010.173

Hybrid superconductor–quantum dot devices

Silvano De Franceschi[1]*, Leo Kouwenhoven[2], Christian Schönenberger[3] and Wolfgang Wernsdorfer[4]

Advances in nanofabrication techniques have made it possible to make devices in which superconducting electrodes are connected to non-superconducting nanostructures such as quantum dots. The properties of these hybrid devices result from a combination of a macroscopic quantum phenomenon involving large numbers of electrons (superconductivity) and the ability to control single electrons, offered by quantum dots. Here we review research into electron transport and other fundamental processes that have been studied in these devices. We also describe potential applications, such as a transistor in which the direction of a supercurrent can be reversed by adding just one electron to a quantum dot.

Quantum dot = artificial atom



single molecule | self-assembled QD | lateral QD | nanotube

1nm | 10nm | 100nm | 1μm

metallic nanoparticle | vertical QD | nanowire



For odd occupancy, it behaves as a magnetic atom.

quantum dot Josephson junction

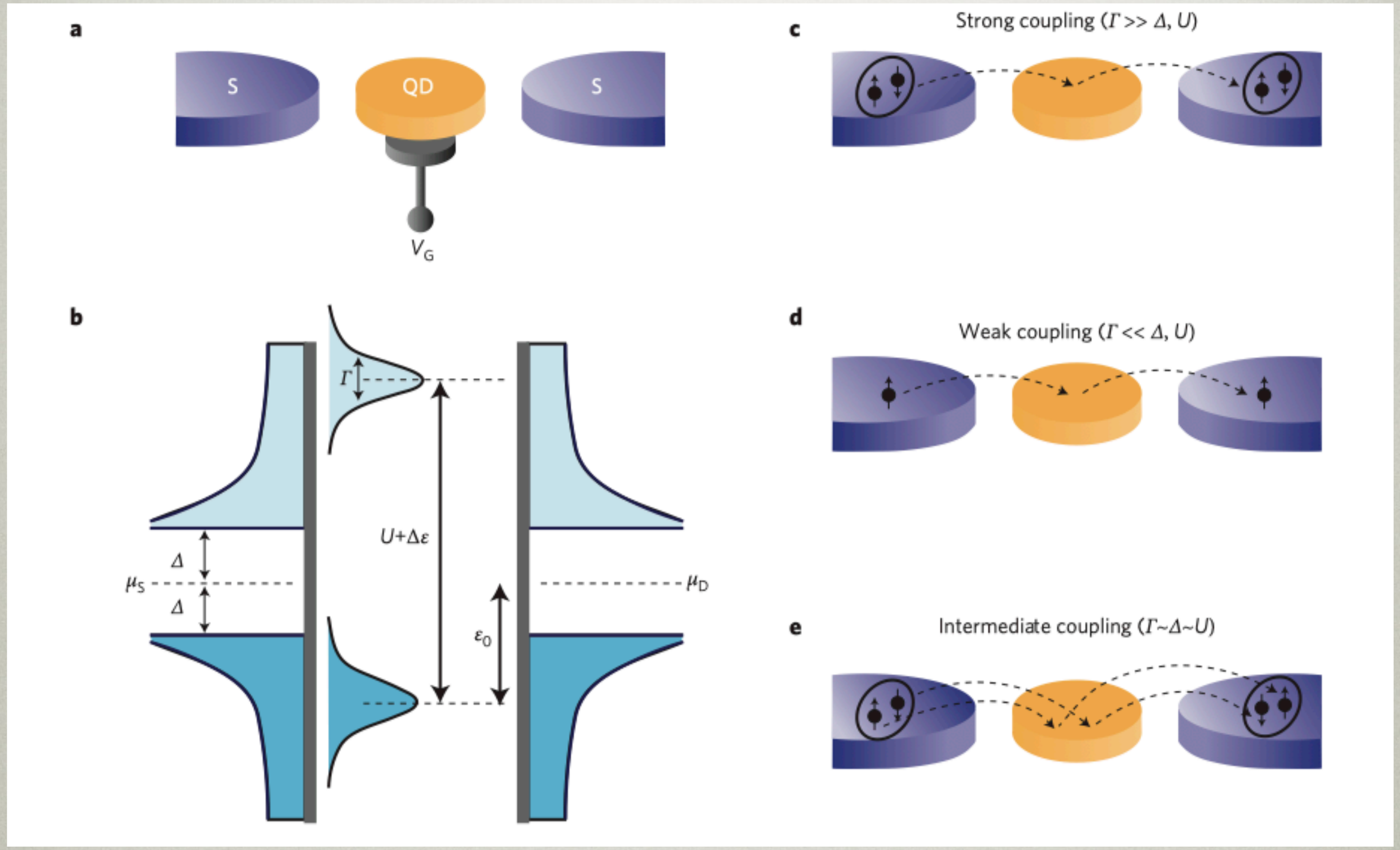
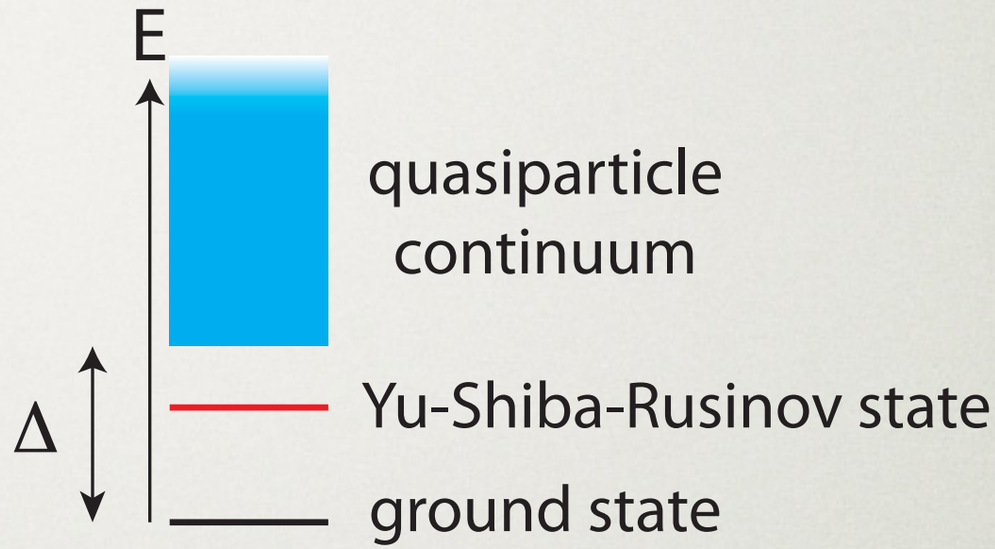


Figure source: de Franceschi et al.

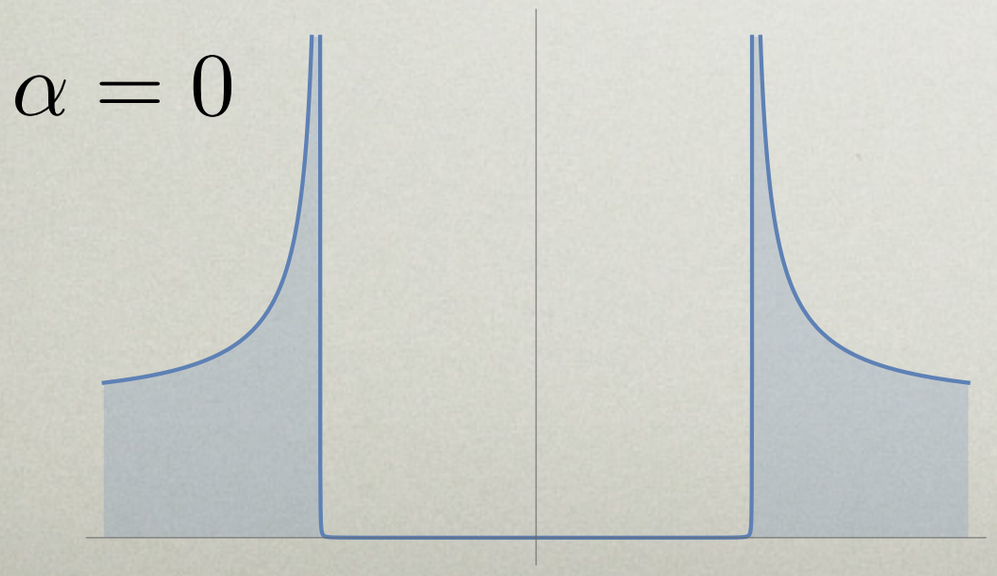
SUBGAP STATES (YU-SHIBA-RUSINOV)

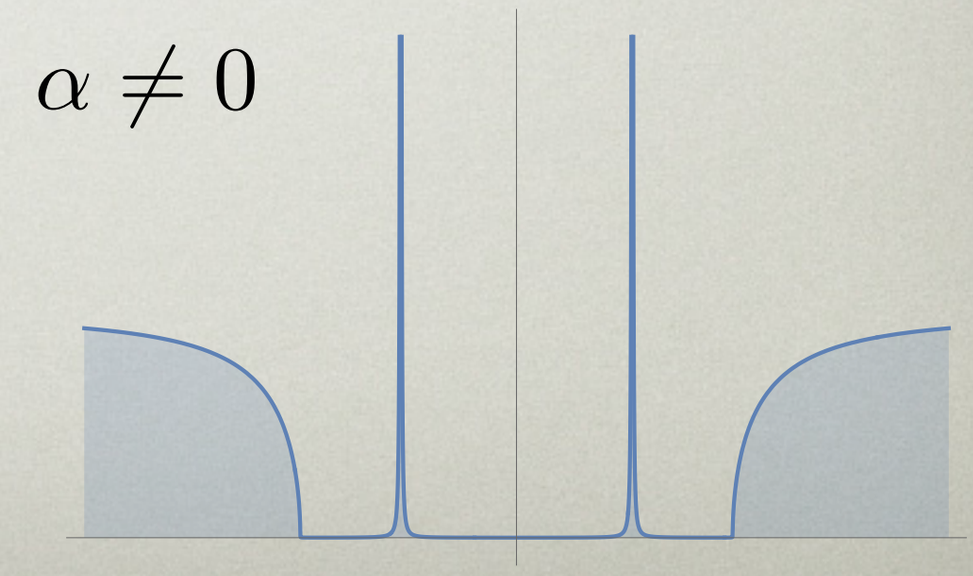
$$E_{\mathrm{YSR}} = \Delta \frac{1-\alpha^2}{1+\alpha^2}$$

$$\alpha = JS\pi\rho/2$$



Yu Lu, Acta Physica Sinica (1965)
H. Shiba, Prog. Theor. Phys. (1968)
A. I. Rusinov, ZhETF Pis. Red (1969)





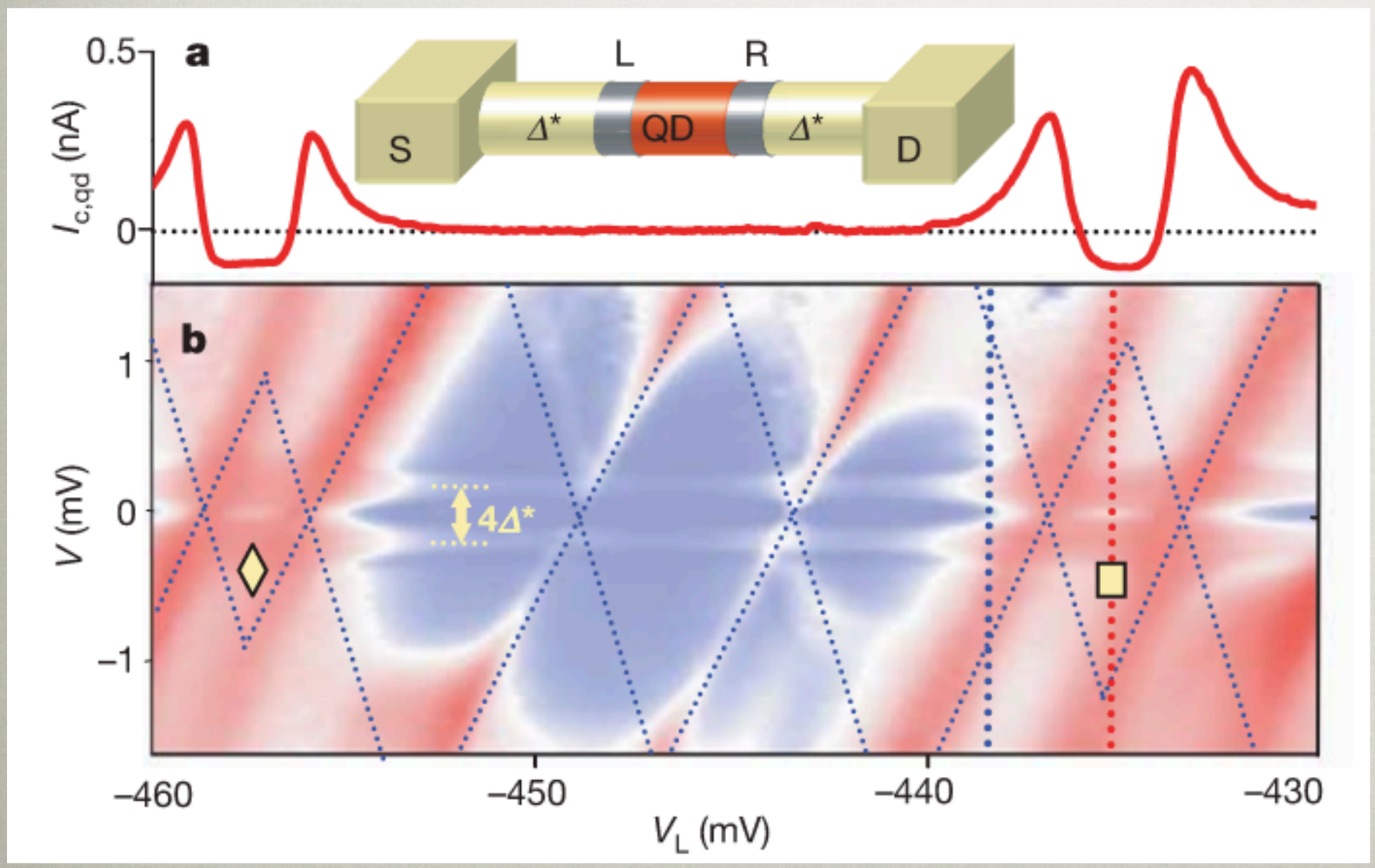
Published: 10 August 2006

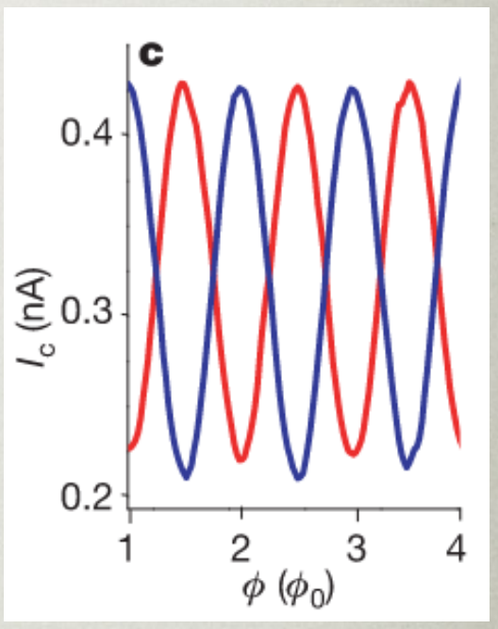
Supercurrent reversal in quantum dots

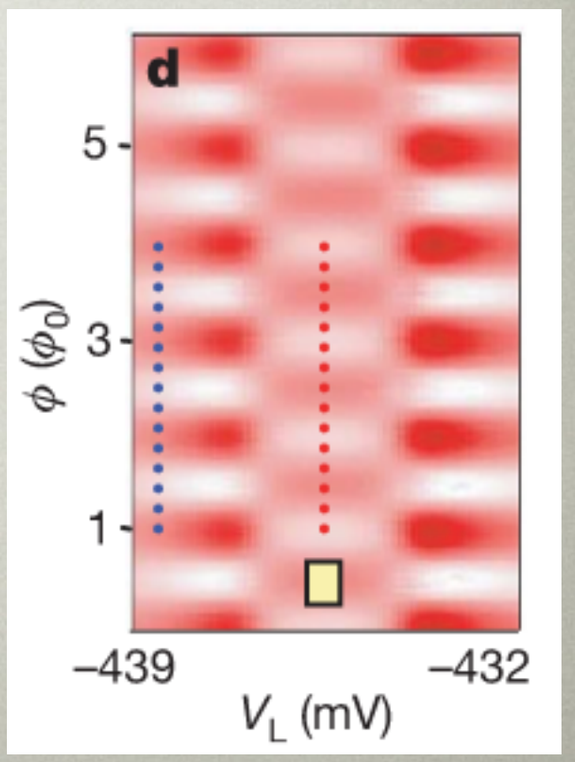
Jorden A. van Dam, Yuli V. Nazarov, Erik P. A. M. Bakkers, Silvano De Franceschi & Leo P. Kouwenhoven

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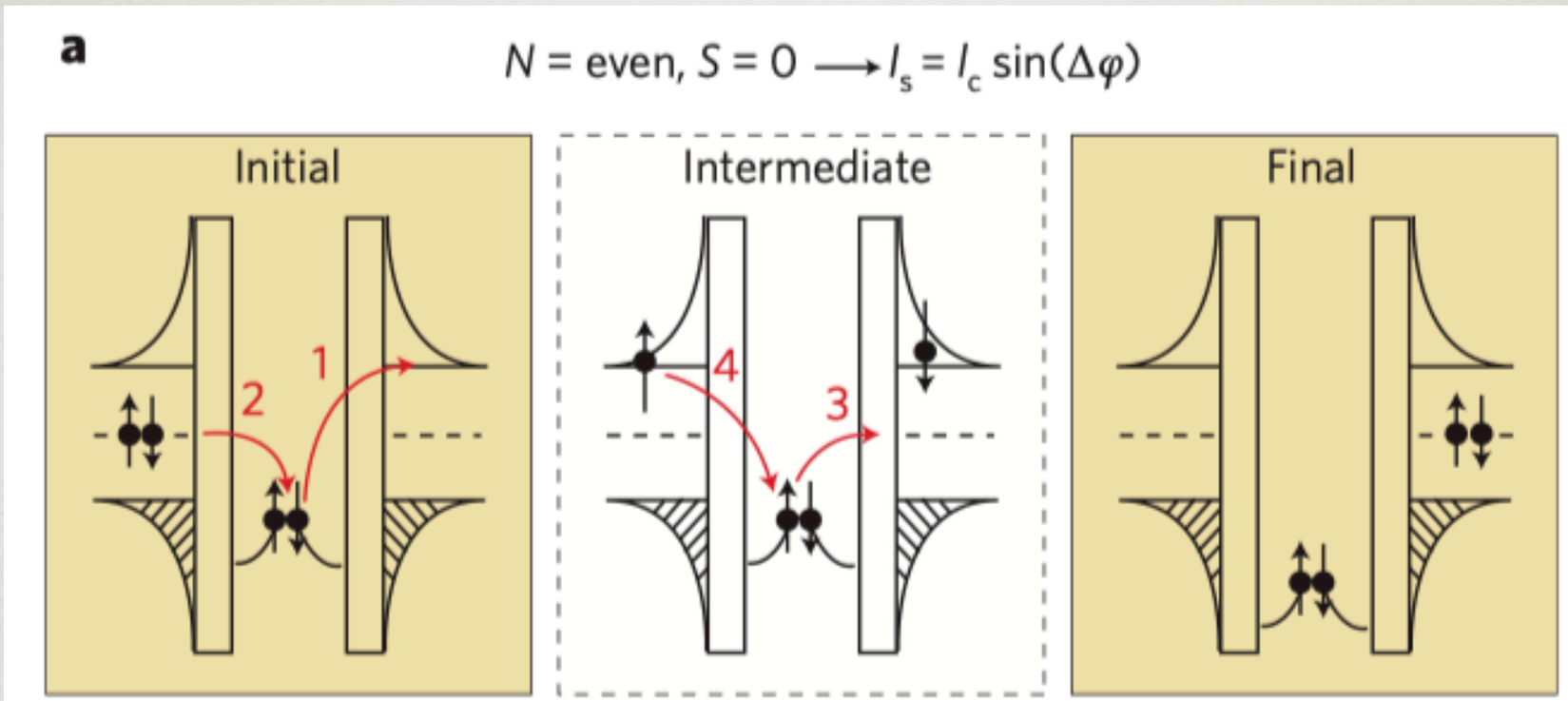
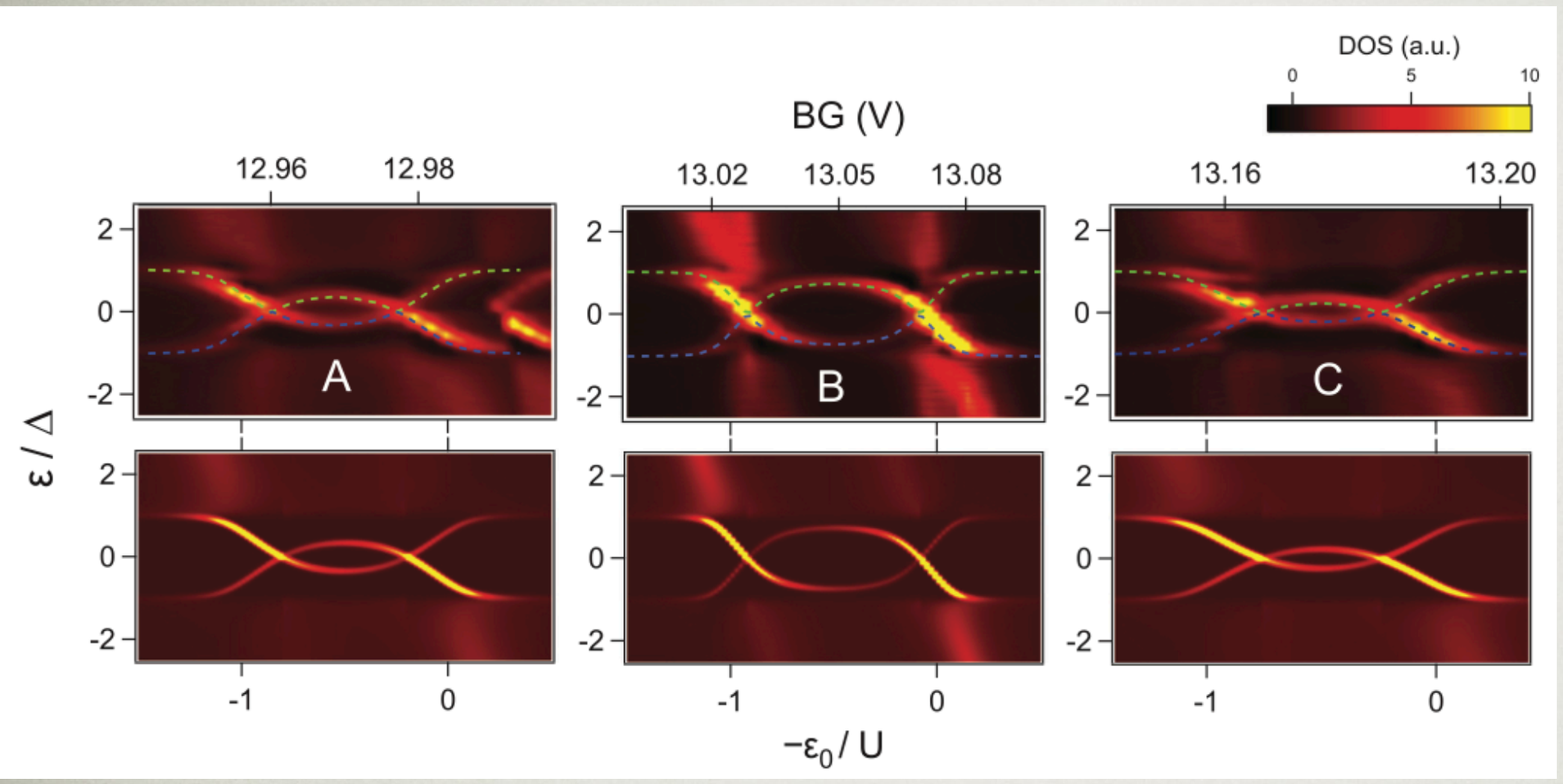


Figure source: de Franceschi et al.

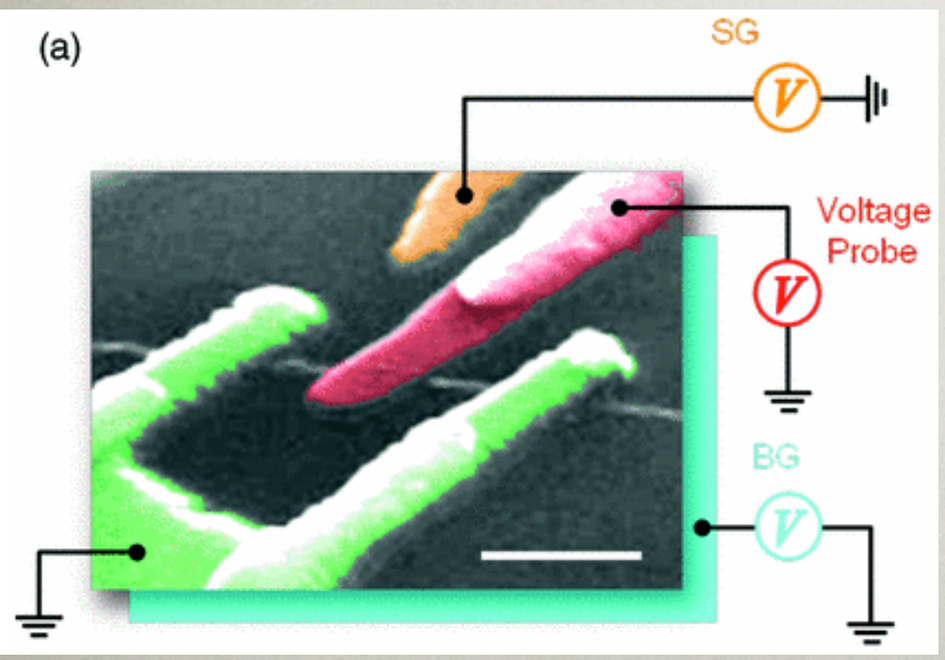
Carbon nanotube quantum dot



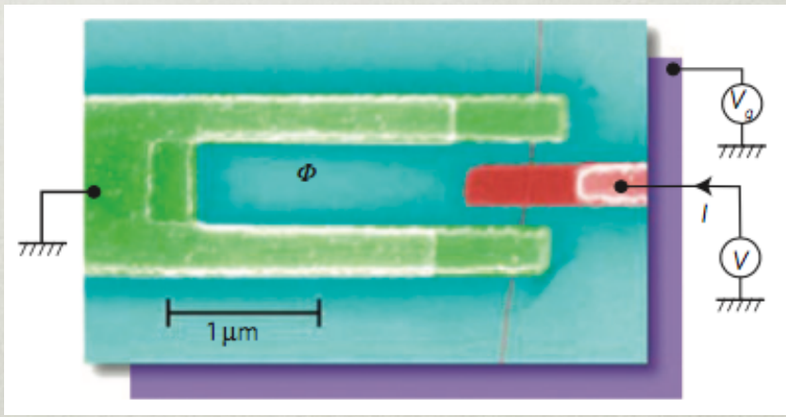
experiment

theory

(NRG calculation)

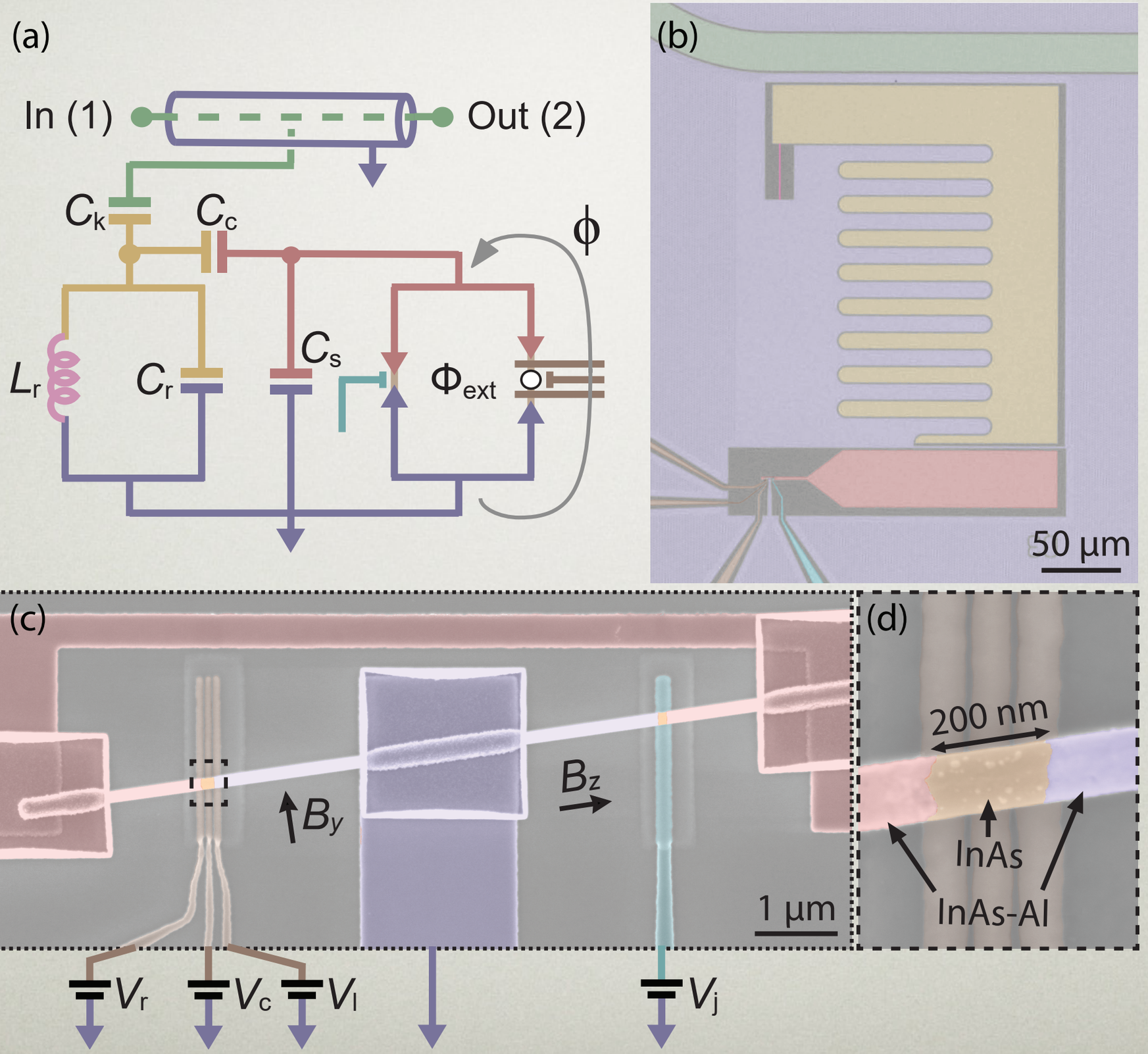


NRG = numerical renormalization group

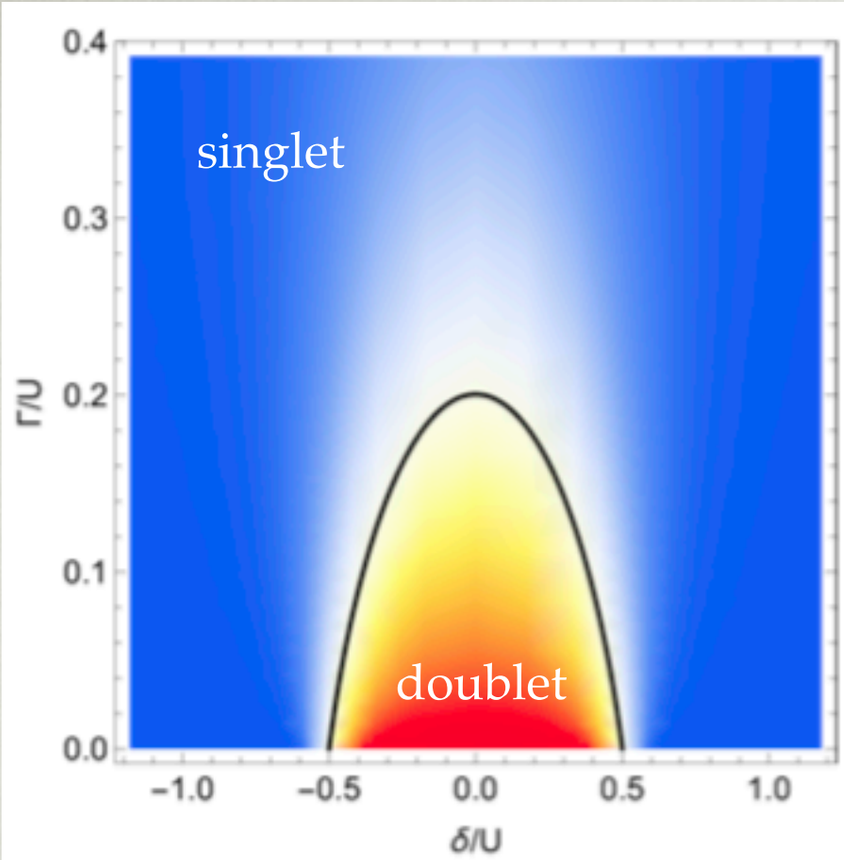


JD Pillet, P Joyez, R Žitko, MF Goffman, PRB 88, 045101 (2013)

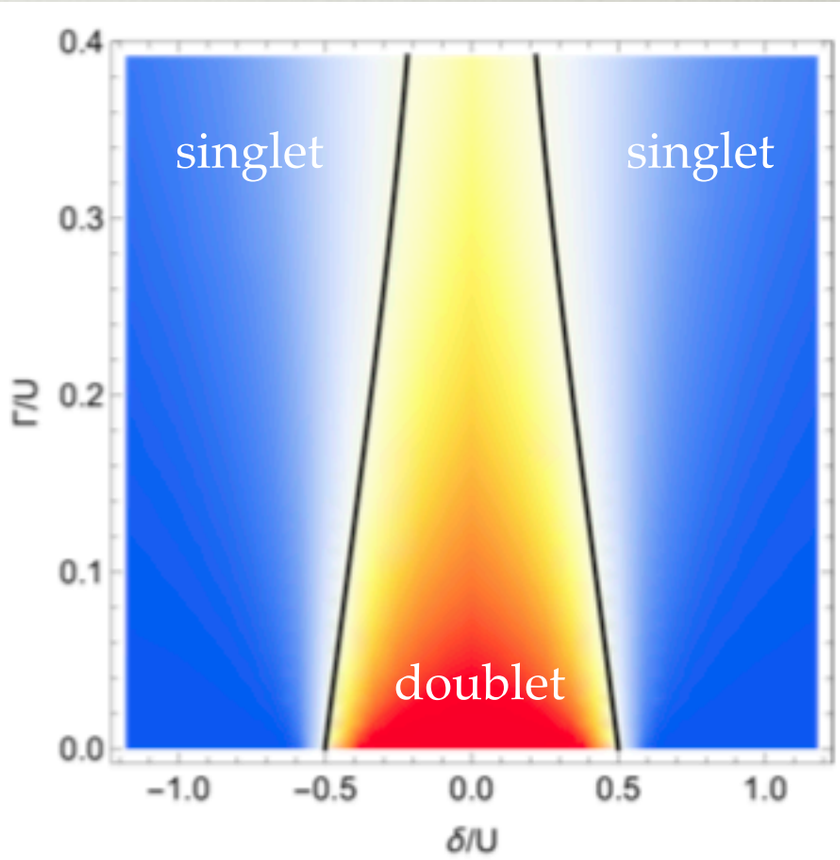
QUANTUM DOTS IN TRANSMONS



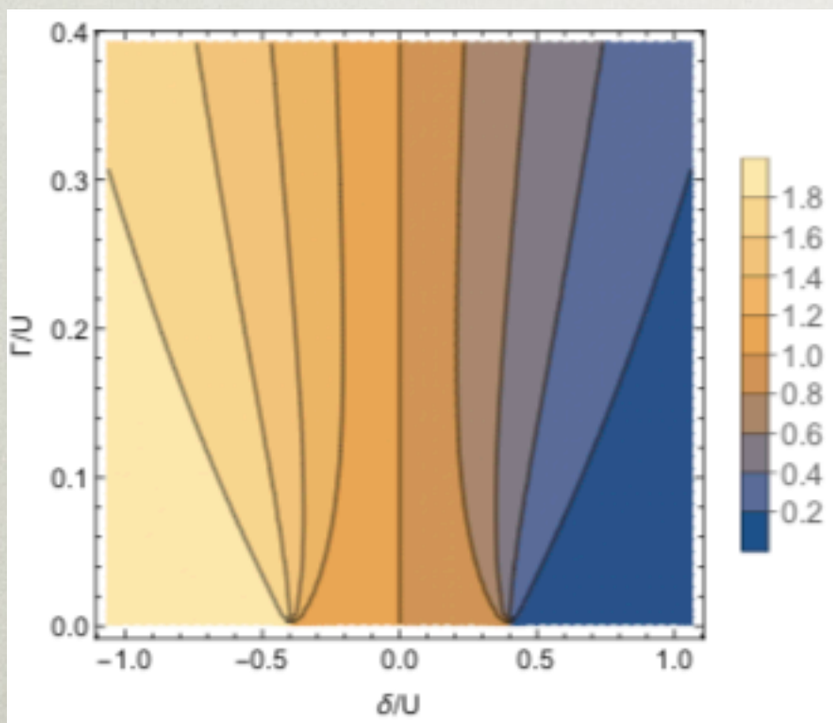
Bargerbos, Pita-Vidal, Žitko, Ávila, Splitthoff, Grünhaupt, Wesdorp, Andersen, Liu, Krogstrup, Kouwenhoven, Aguado, Kou, van Heck (2022)



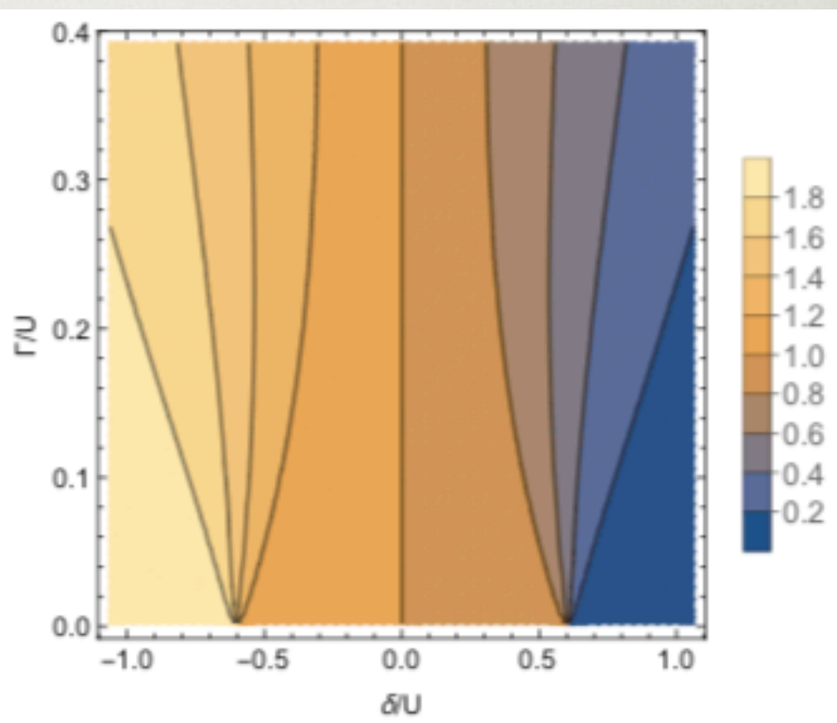
(a) Phase diagram at $\phi = 0$



(b) Phase diagram at $\phi = \pi$

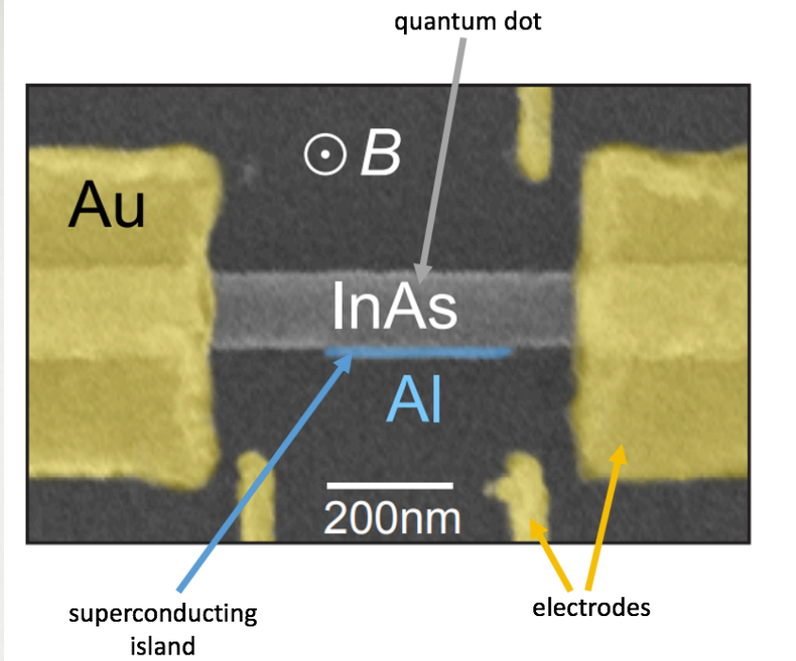


(g) n_{imp} for $S = 0$



(h) n_{imp} for $S = 1/2$

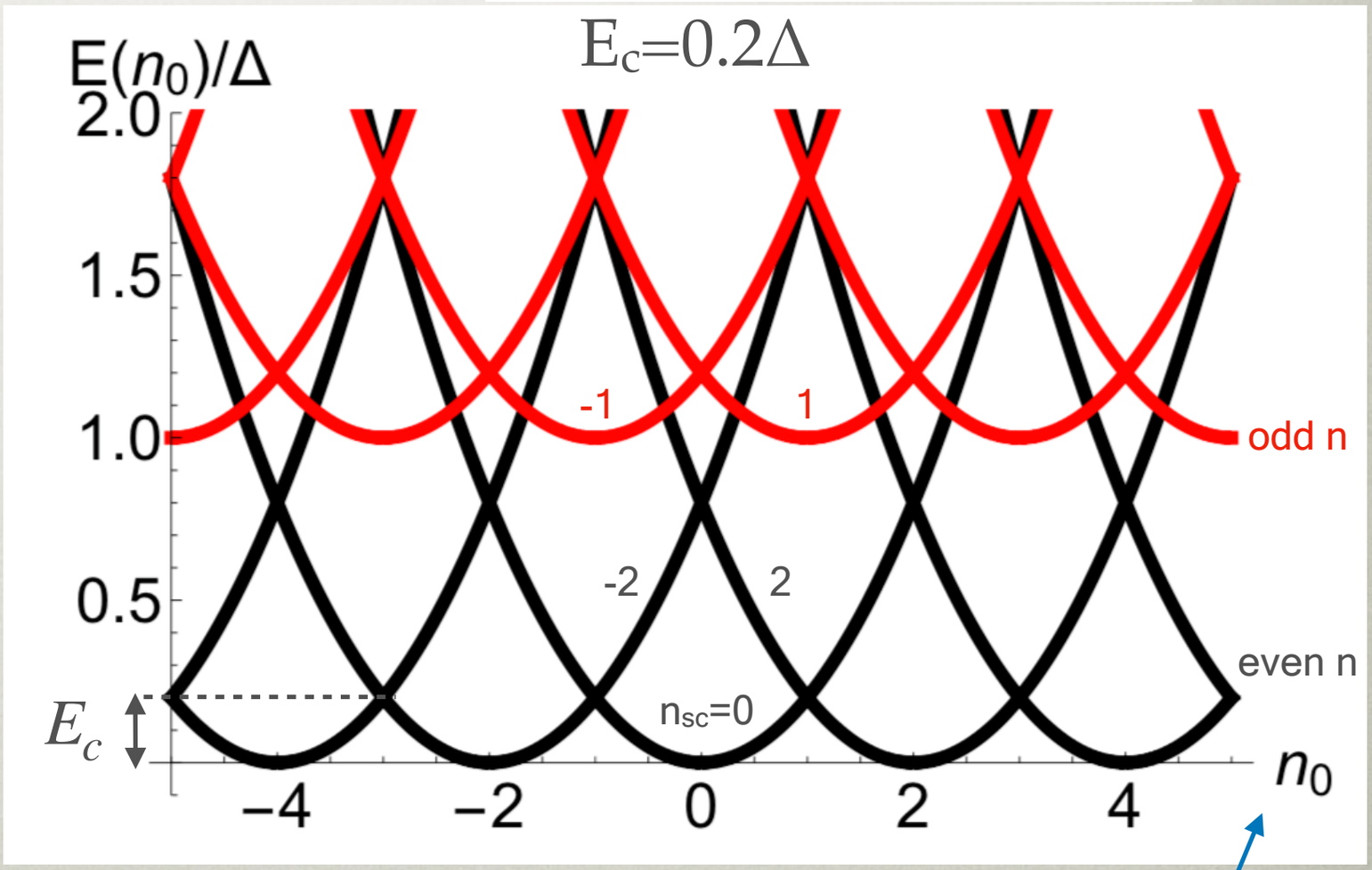
Superconducting islands



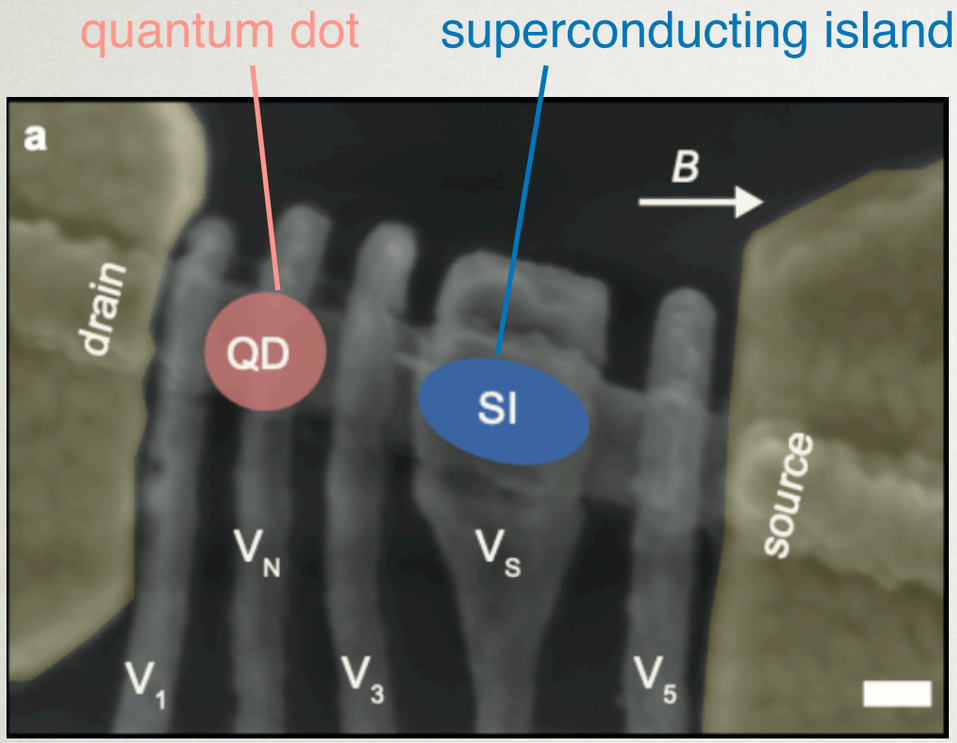
A. P. Higginbotham et al., 2015

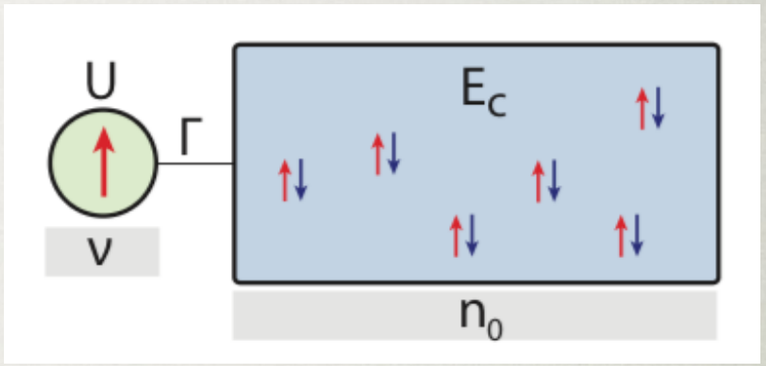
$$E_c(\hat{n}_{\mathrm{sc}} - n_0)^2$$

$$E_c = \frac{e_0^2}{2C}$$ charging energy



energy-level / gate-voltage expressed in units of electron number

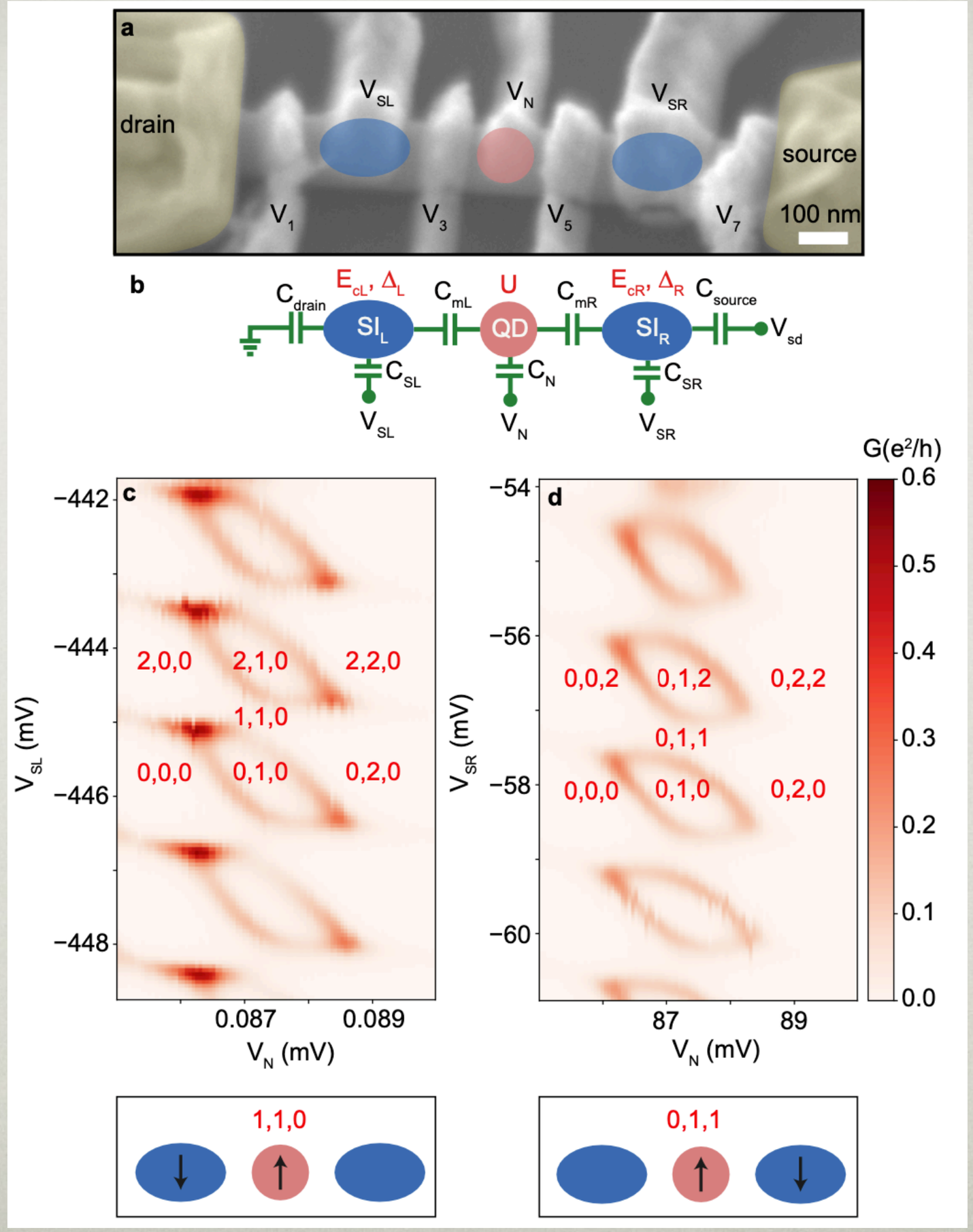




$$H = \sum_i \epsilon_i c^\dagger_{i\sigma} c_{i\sigma} - \alpha d \sum_{i,j} c^\dagger_{i\uparrow} c^\dagger_{i\downarrow} c_{j\downarrow} c_{j\uparrow} \quad + \quad E_c(\hat{n}_{\mathrm{sc}} - n_0)^2$$

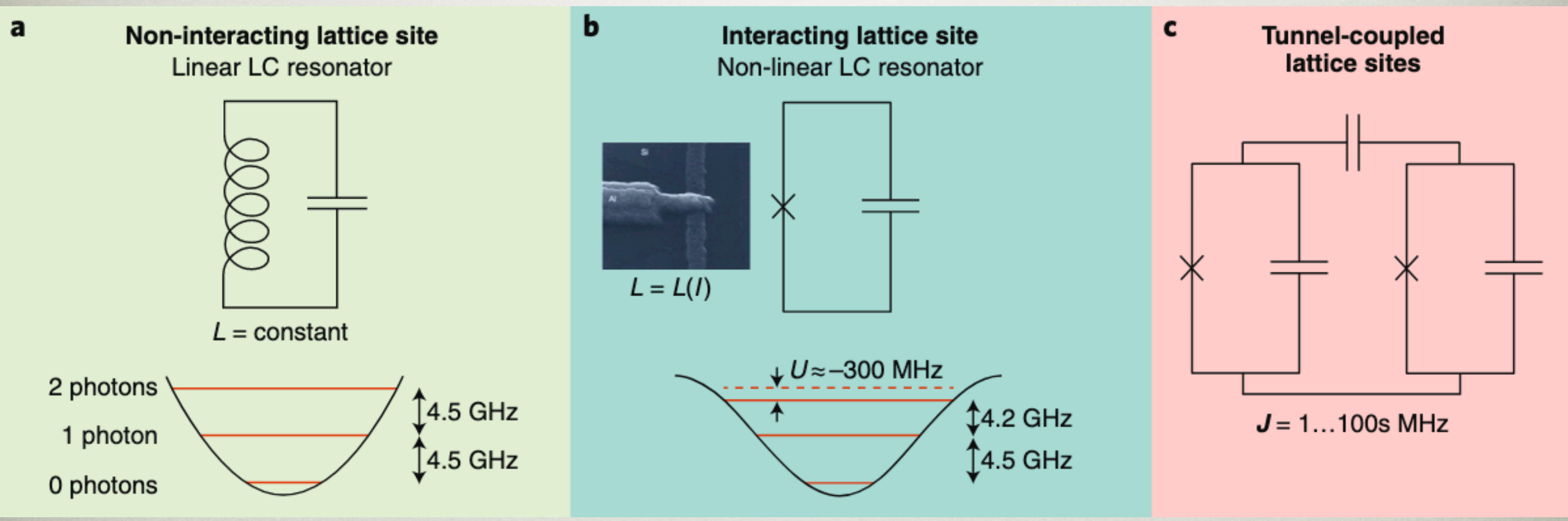
Richardson model + charge repulsion

Pavešić, Bauernfeind, Žitko, PRB 2021
Pavešić, Žitko, PRB 2022
J. C. Estrada Saldaña et al., arxiv:2101.10794



J. C. Estrada Saldaña et al., arxiv:2101.10794

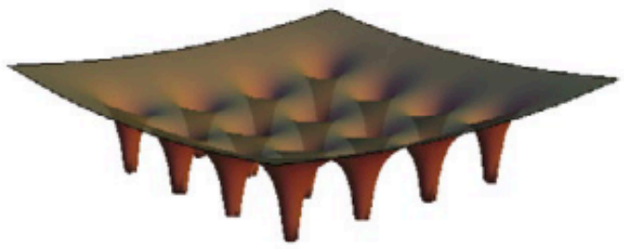
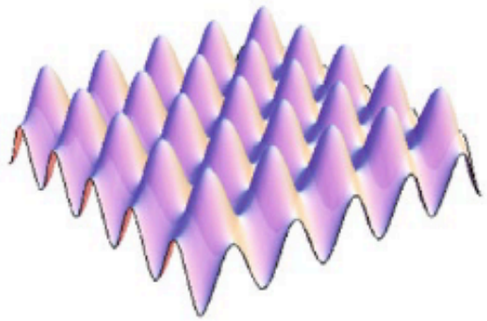
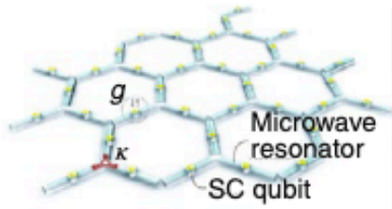
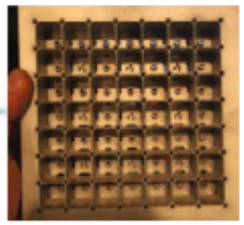
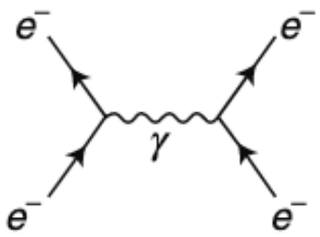
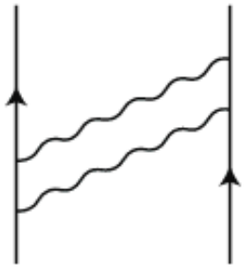
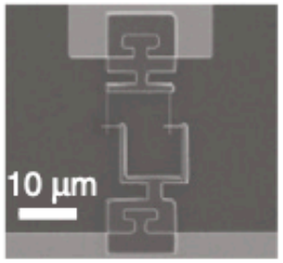
Perspectives



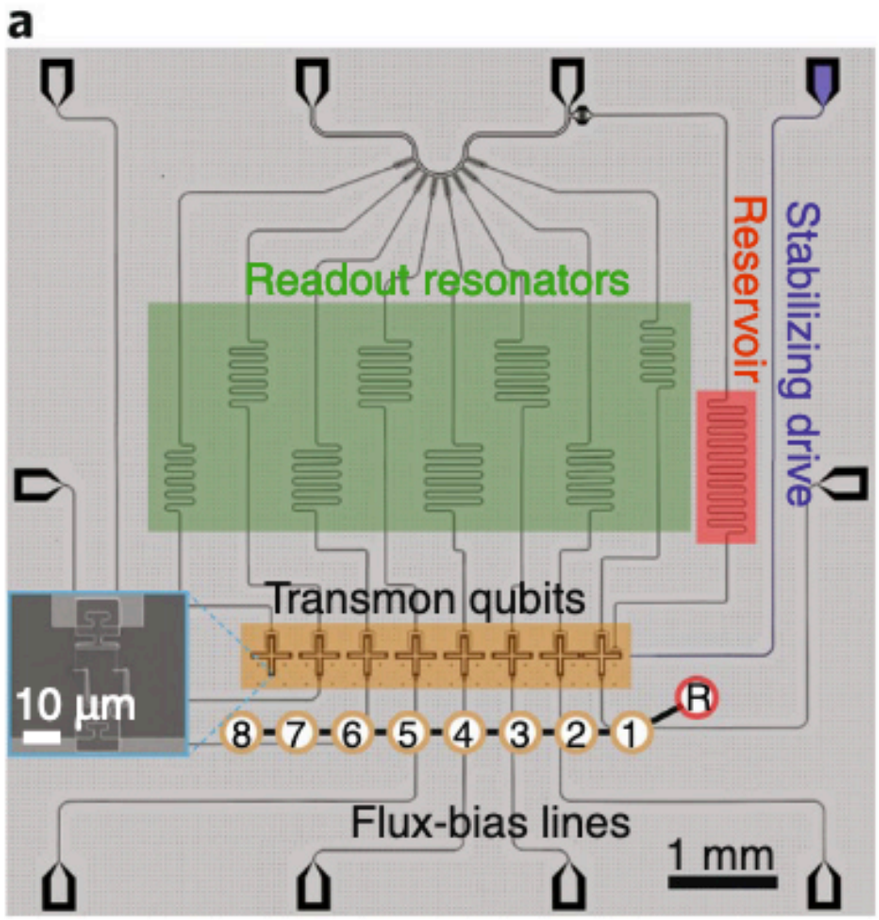
Source: Carusotto et al. RMP (2020)

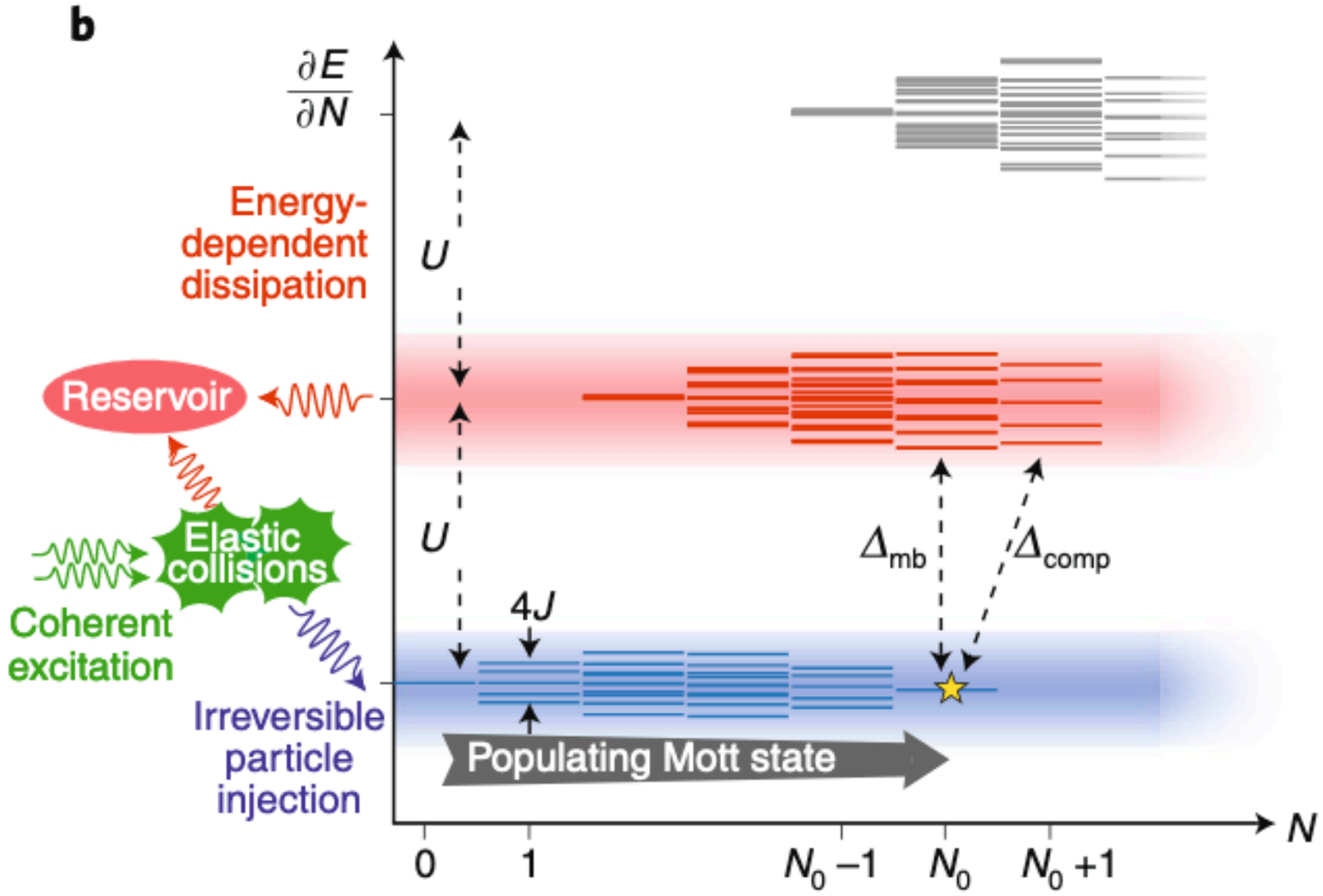
Quantum simulators

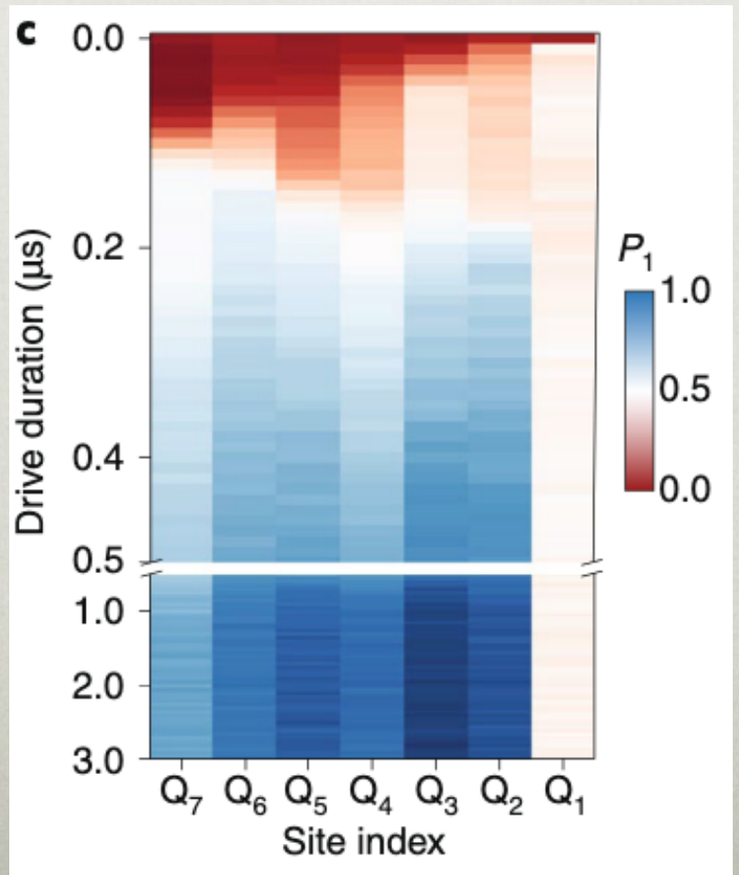
Quantum matter platforms

	2D electron gas	Ultracold atoms	Microwave photons
Trapped in:			 
	Ionic lattice	Optical lattice	Circuits and meta-materials
Interactions mediated by:			
	Coulomb potential	Van der Waals potential	Transmon qubits
Interaction energy scale	80 K ~ h x 2 THz	20 nK ~ h x 400 Hz	10 mK ~ h x 200 MHz
Coherence time	2 ns	10 s	100 µs
Interaction to coherence ratio	~3,000	~4,000	~20,000
Strength of the platform	Direct application in real world, clean-room fabrication	Scalable, optical manipulation/readout	Arbitrary connectivity, reservoir engineering

Source: Carusotto et al. RMP (2020)

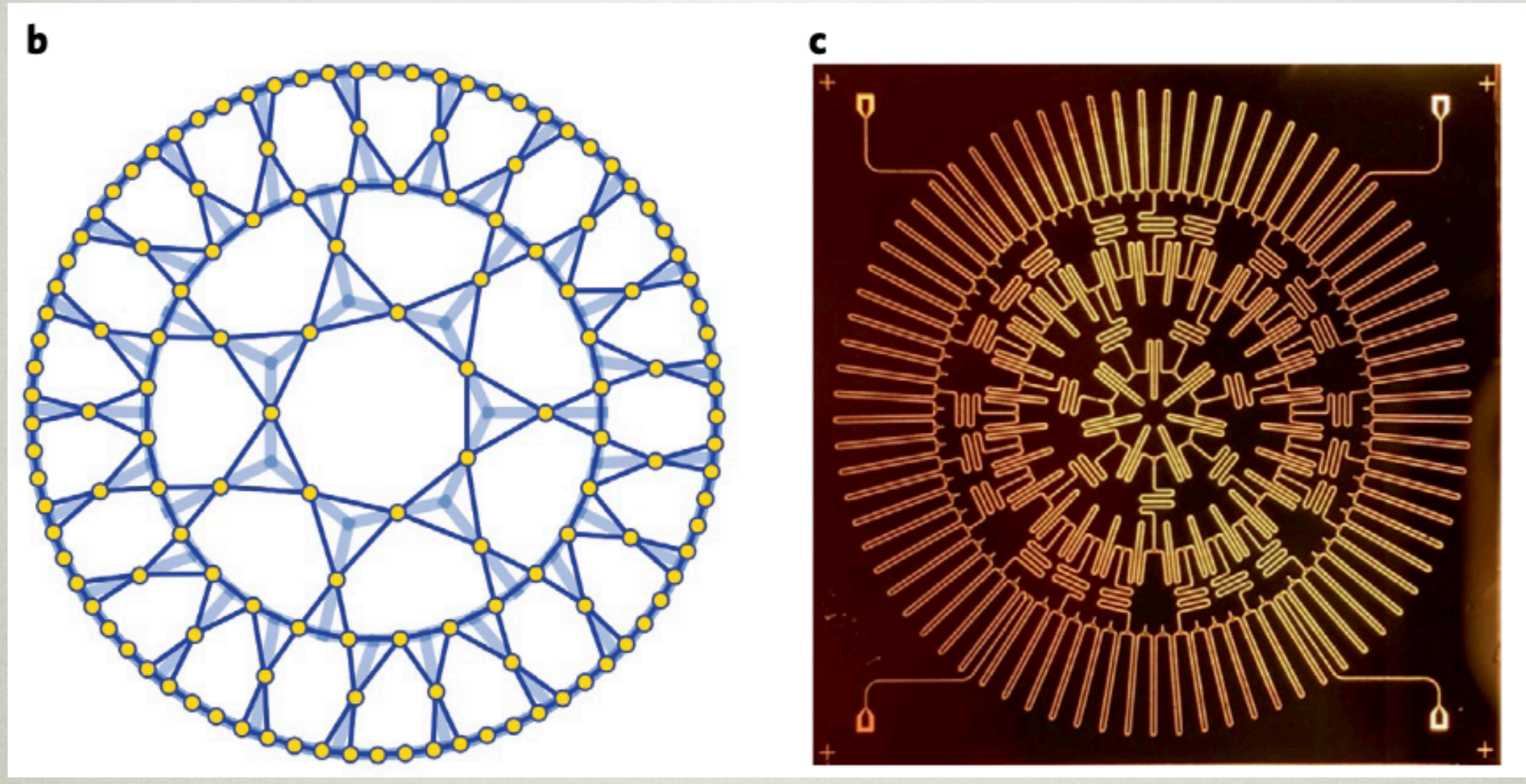






Source: Carusotto et al. RMP (2020)

Simulations of curved space-time



Source: Carusotto et al. RMP (2020)

Literature

a) Devoret: Quantum fluctuations in electrical circuits, Les Houches, Session LXIII, 1995, Elsevier (1997), https://boulderschool.yale.edu/sites/default/files/files/devoret_quantum_fluct_les_houches.pdf updated version https://arxiv.org/pdf/1610.03438.pdf
b) Blais, Grimsmo, Girvin, Wallraff, Circuit Quantum Electrodynamics, Rev. Mod. Phys. 93, 25005 (2021), https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.93.025005
c) Blais, Girvin, Oliver: Quantum information processing and quantum optics with circuit quantum electrodynamics, Nature Physics 16, 247 (2020), https://www.nature.com/articles/s41567-020-0806-z
d) Blais, Huang, Wallraff, Girvin, Schoelkopf: Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation, Phys Rev A 69, 062320 (2004), https://journals.aps.org/pra/abstract/10.1103/PhysRevA.69.062320
e) Martinis, Devoret, Clarke, Quantum Josephson junction circuits and the dawn of artificial atoms, Nature Physics 16, 234 (2020), https://www.nature.com/articles/s41567-020-0829-5