Numerical renormalization group
part 1

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Uberlandia school of advanced studies, January 2019

Quantum impurity

- point-like object with a local degree of freedom
  example: spin & orbital angular momentum of a magnetic dopant atom
- environment = continuum of states
  example: itinerant electrons in a metal
- interaction that can change the value of local d.o.f.
  example: exchange coupling & spin-flip scattering

SPIN-BOSON MODEL



$|0\rangle$



$|1\rangle$

two-level system, “spin”



continuum (sea) of levels

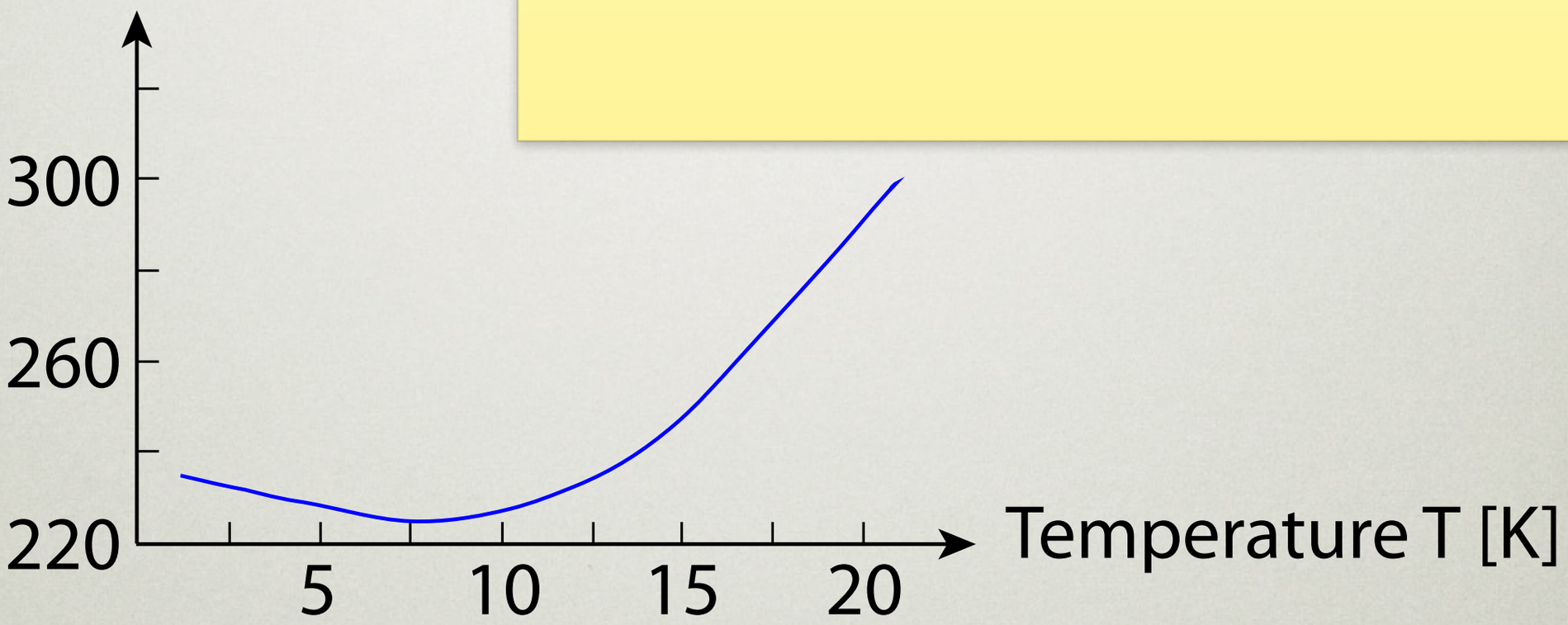
$$H = \Delta\sigma_x + \sum_i \left(\frac{p_i^2}{2m} + \frac{1}{2} m\omega_i^2 x_i^2 \right) + \sigma_z \sum_i c_i x_i$$

model for studying decoherence and quantum dissipation

Magnetic impu

Low-temperatu

Solid state physics text books tell us that Matthiessen's rule is obeyed by simple metals: the temperature dependent resistivity is the sum of a temperature independent term due to elastic scattering off impurities and an impurity independent term which is temperature dependent due to inelastic scattering.
http://condensedconcepts.blogspot.com/2012/01/violating-text-book-rule.html



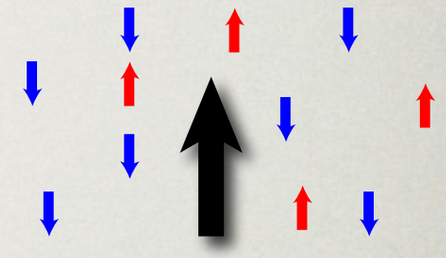
resistance minimum at finite temperature

Minimum depends on impurity concentration.

W. J. de Haas and G. J. van den Berg, Physica **3**, 440 (1936)

Kondo model

$$H = H_{\text{band}} + H_{\text{exch}}$$



$$H_{\text{band}} = \sum_{\mathbf{k},\sigma} \epsilon_k \, c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma}$$

Fermi sea: gas of non-interacting electrons

$$H_{\text{exch}} = J \, \mathbf{S} \cdot \mathbf{s}$$

exchange coupling

$\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ quantum-mechanical spin operator

$\mathbf{s} = \frac{1}{N} \sum_{kk'} c^\dagger_k \left(\frac{1}{2}\boldsymbol{\sigma}\right) c_{k'} = f^\dagger_0 \left(\frac{1}{2}\boldsymbol{\sigma}\right) f_0$ spin-density (at $\mathbf{r}$=0)

(Zener, 1951)

Quantum-mechanical spin

$$[S^i, S^j] = i\epsilon_{ijk} S^k$$

$$\mathbf{S}^2 |s, m\rangle = s(s+1)|s, m\rangle$$

$$S^z |s, m\rangle = m|s, m\rangle$$

$$S^+ |s, m\rangle = \sqrt{s(s+1) - m(m+1)}|s, m+1\rangle$$

$$S^- |s, m\rangle = \sqrt{s(s+1) - m(m-1)}|s, m-1\rangle$$

$$S^+ = S^x + iS^y$$

$$S^- = S^x - iS^y$$

Spin-1/2

$$\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\mathbf{S} = \frac{1}{2}\begin{pmatrix} d_\uparrow^\dagger & d_\downarrow^\dagger \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} d_\uparrow \\ d_\downarrow \end{pmatrix}$$

$$S^+ = d_\uparrow^\dagger d_\downarrow \qquad S^- = d_\downarrow^\dagger d_\uparrow \qquad S^z = \frac{1}{2}\left(d_\uparrow^\dagger d_\uparrow - d_\downarrow^\dagger d_\downarrow\right)$$

Exchange interaction

$$\mathbf{S} \cdot \mathbf{s} = S^z s^z + \frac{1}{2}\left(S^+ s^- + S^- s^+\right)$$

spin-preserving scattering

spin-flip scattering

Impurity contribution to ...

How does the impurity change the properties of the system?

$$\langle o \rangle_{\text{imp}} = \langle o \rangle - \langle o \rangle_0$$

System with impurity

Clean system

$$\langle O \rangle = \langle O \rangle_0 + Nc\langle o \rangle_{\text{imp}} + \mathcal{O}(Nc^2)$$

This is known as the “dilute limit”.

NON-PERTURBATIVE BEHAVIOR

$$H' = -\boldsymbol{\mu} \cdot \mathbf{B} = g\mu_B S_z B$$

$$M_{\text{imp}} = g\mu_B \left(\langle S_z + S_{z,\text{band}} \rangle - \langle S_{z,\text{band}} \rangle_0 \right)$$

$$\chi_{\text{imp}}(T) = \frac{(g\mu_B)^2}{k_B T} \frac{1}{4} \left[1 - \rho J + (\rho J)^2 \ln(k_B T / D) + \ldots \right]$$

$$R(T) = aT^5 + c_{\text{imp}} R_0 - c_{\text{imp}} R_1 \ln\left(\frac{k_B T}{D} \right)$$

Kondo (1964)

Resummation to infinite order does not remove the divergence.

Kondo problem

Perturbation theory breaks down for arbitrarily small J at low enough temperatures.

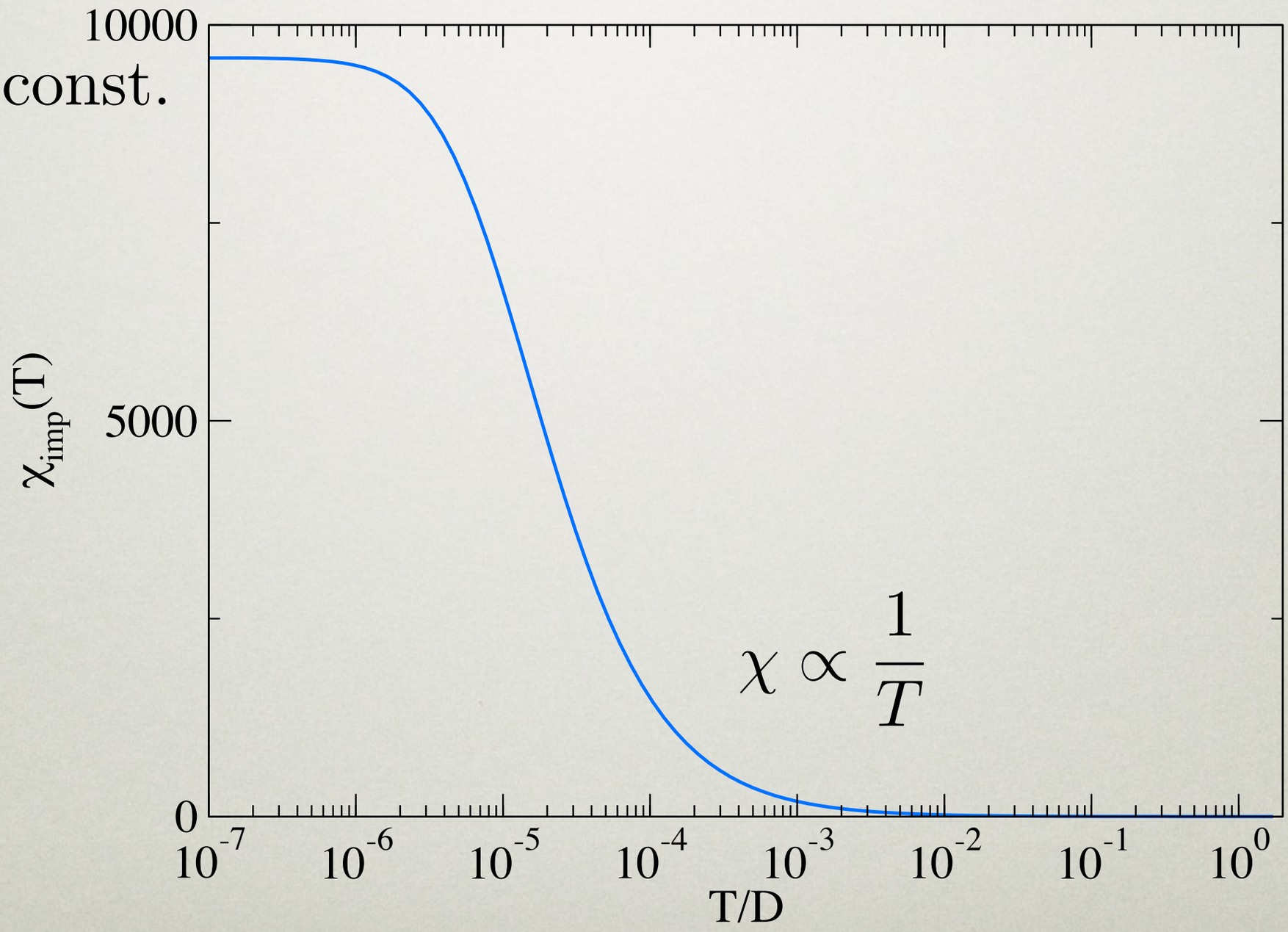
$$k_B T_K \sim D e^{-1/\rho J}$$

Kondo temperature

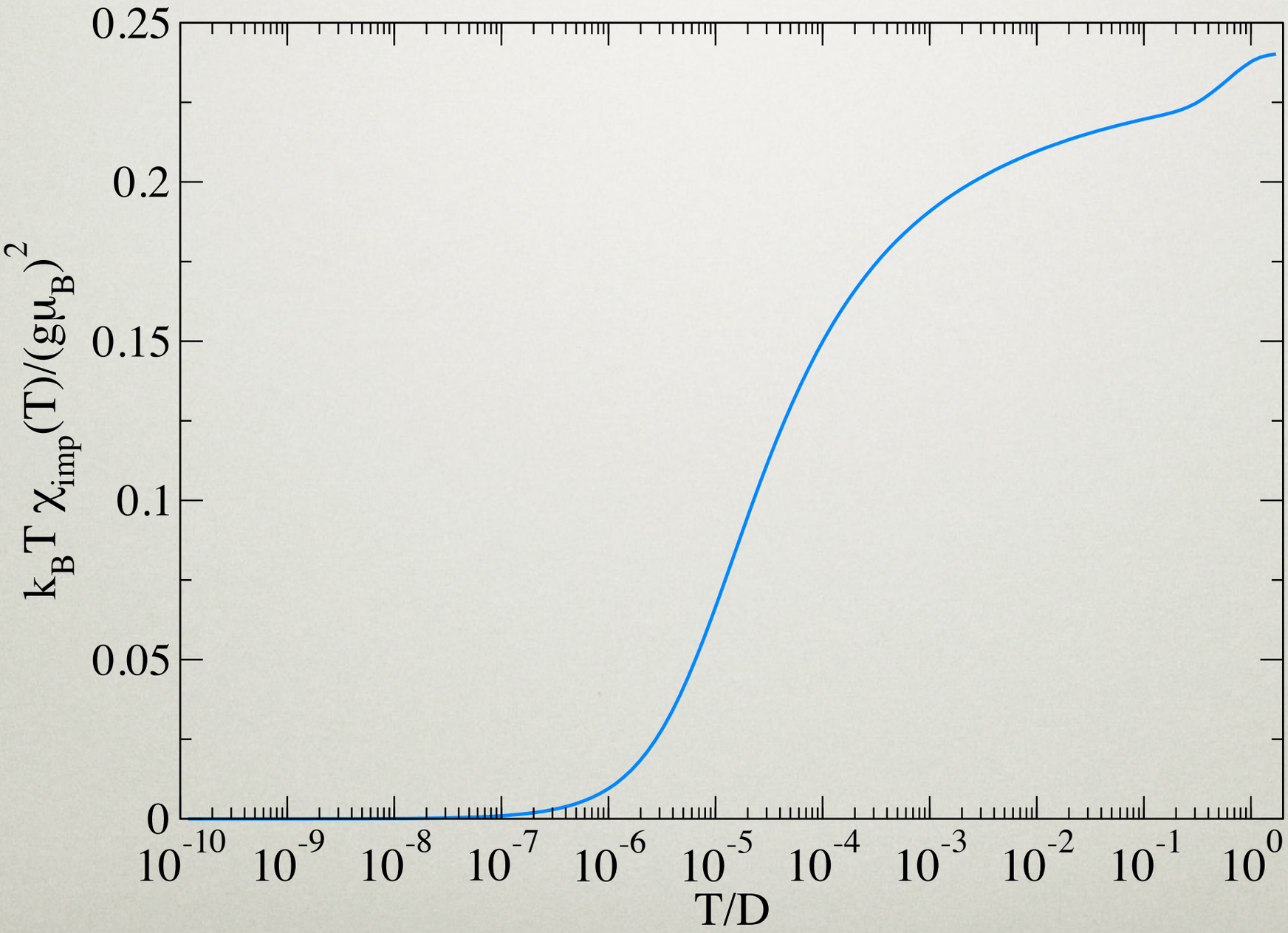
The Kondo problem: finding a solution valid in the low-temperature $T<T_K$ regime.

Saturation of impurity susceptibility

$$\chi \propto \frac{1}{T_K} = \text{const.}$$

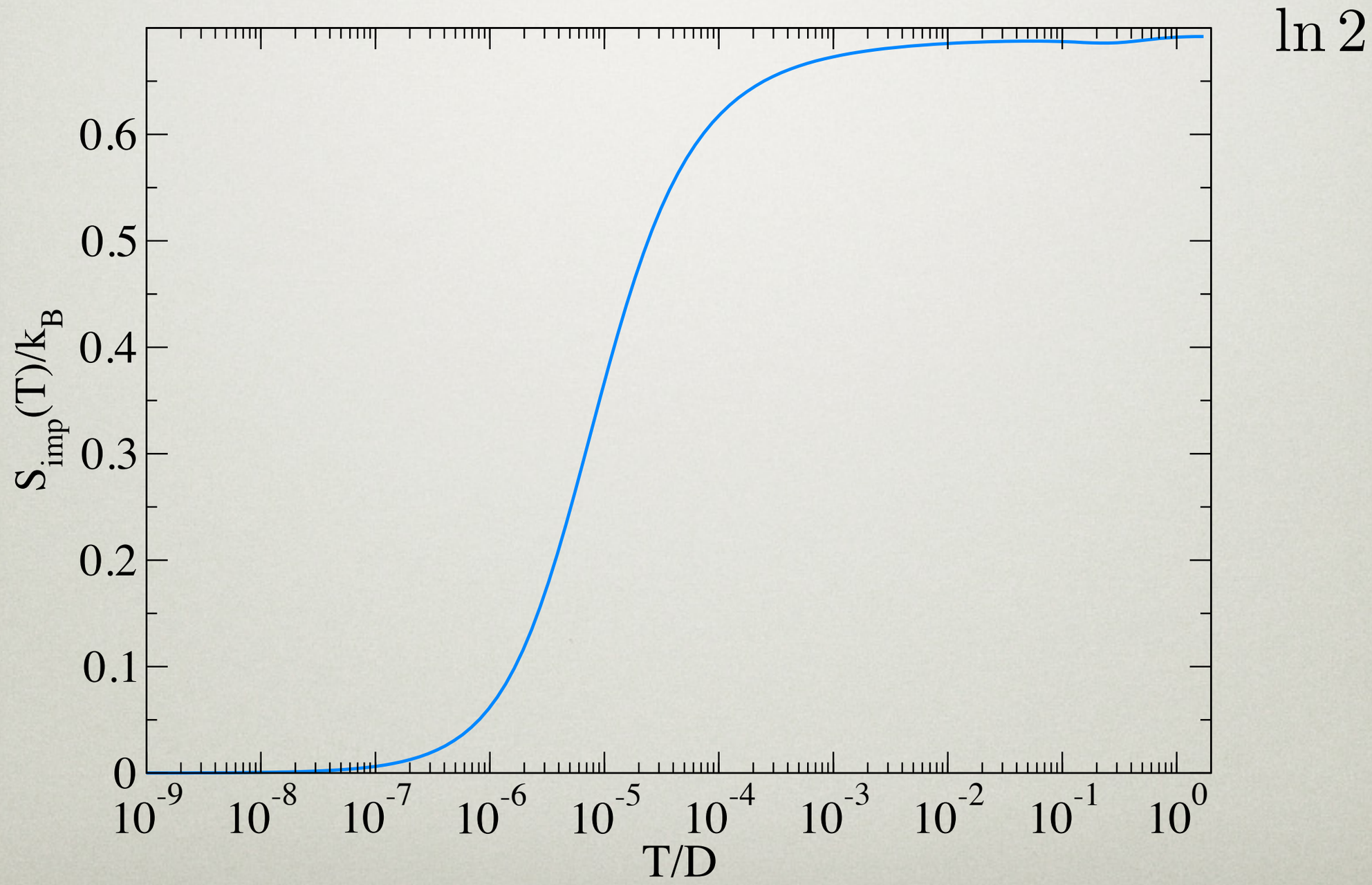


Screening of the local moment

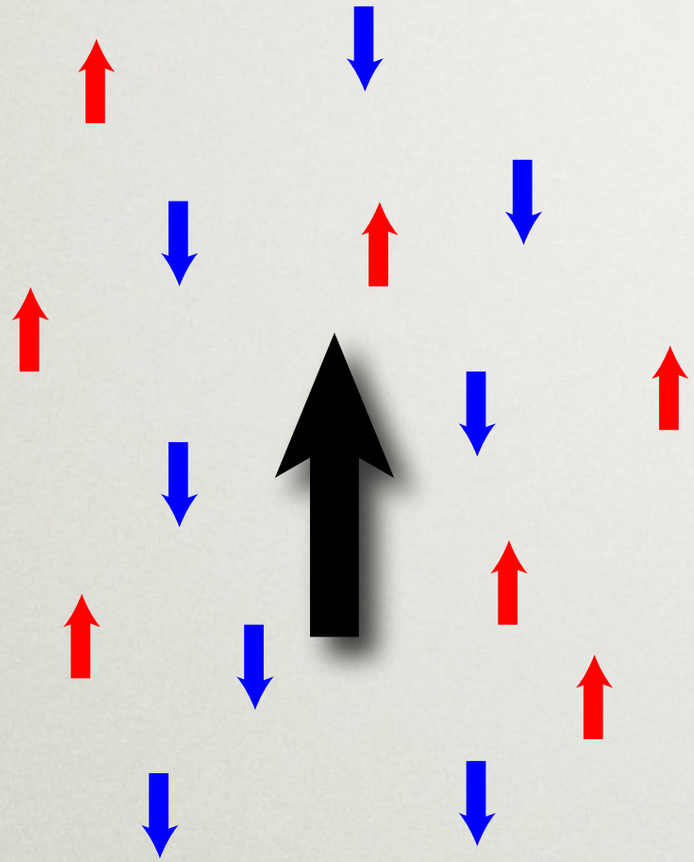


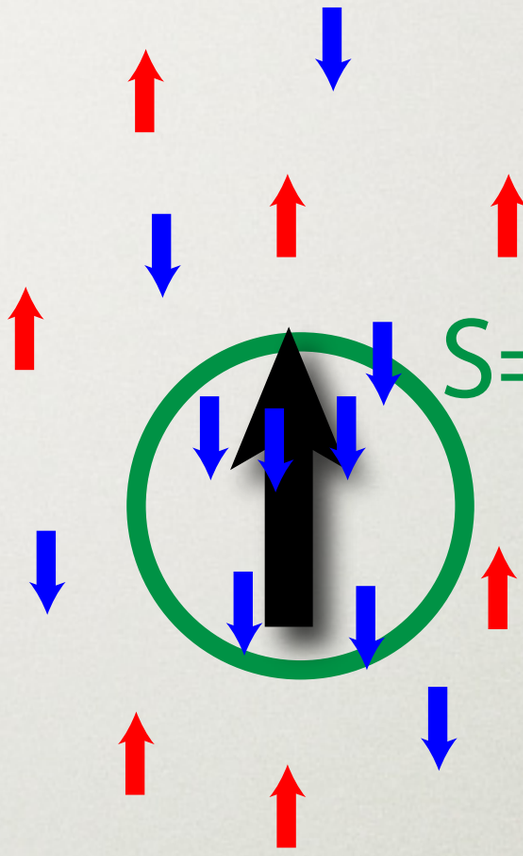
This implies the ground state is a **spin singlet**.

Reduction of impurity entropy



the Kondo effect





Kondo singlet

$$T_K = \mathcal{D} \exp\left(-\frac{1}{\rho J}\right)$$

$T \gg T_K$

free local moments

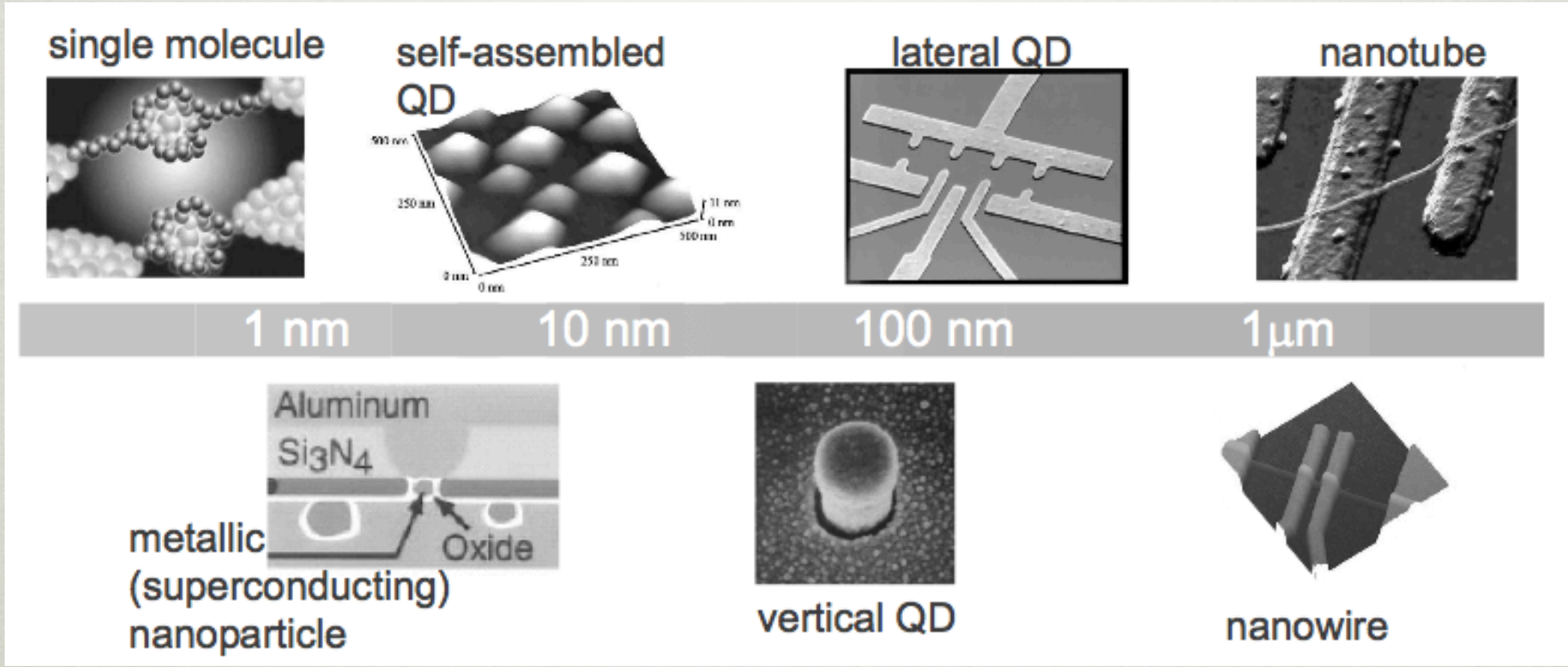
"asymptotic freedom"

$T \ll T_K$

screened local moments

"infrared slavery"

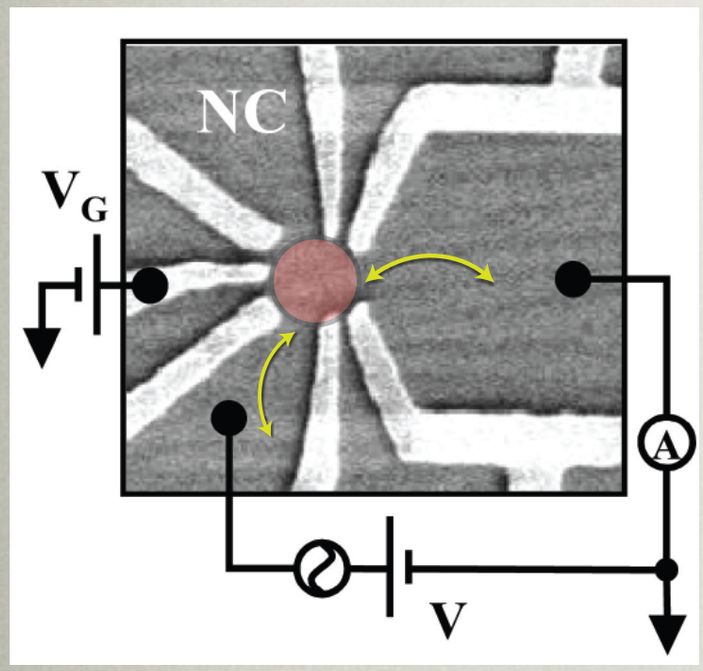
Quantum dots (=artificial atoms)



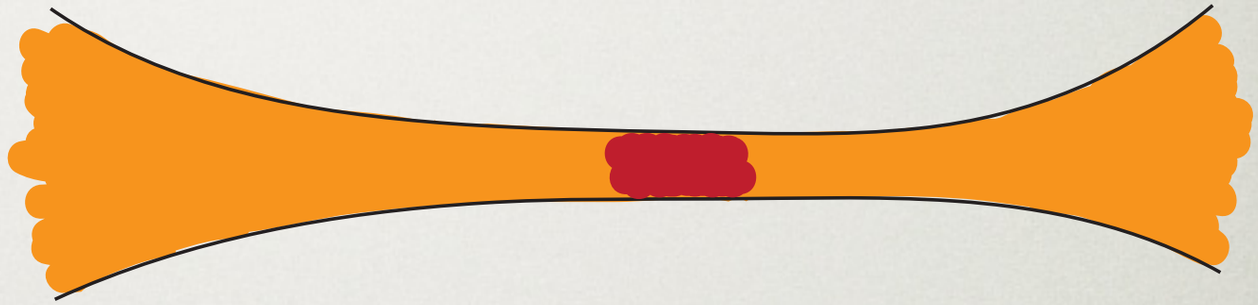
"tunable atoms"

magnetic for odd number of trapped electrons

Transport in nanostructures



Grobis et al., PRL 100, 246601 (2008)



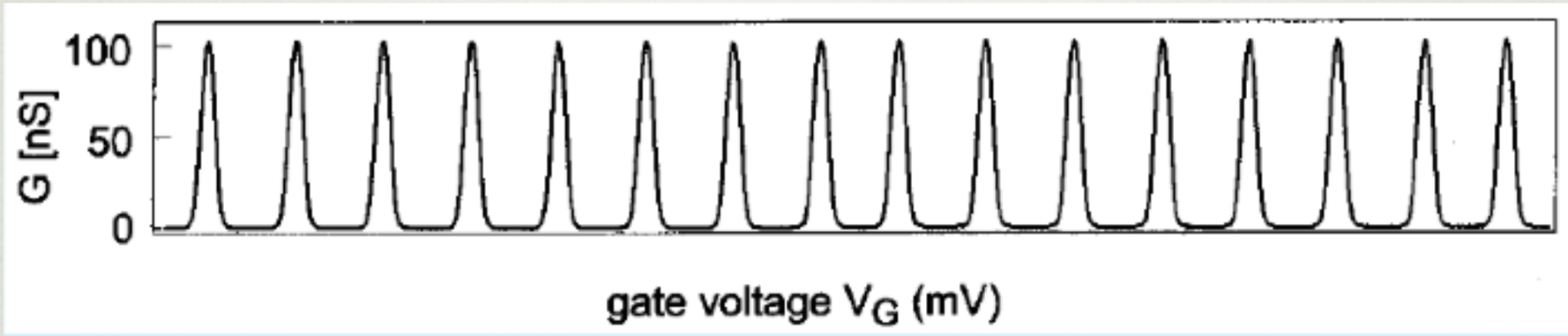
transmission coefficient, T(ϵ)

Landauer formula: $G = \frac{e^2}{h} \sum_\sigma T_\sigma(E_F)$

conductance quantum: $G_0 = \frac{2e^2}{h} = 1/12.906\,\mathrm{k}\Omega$

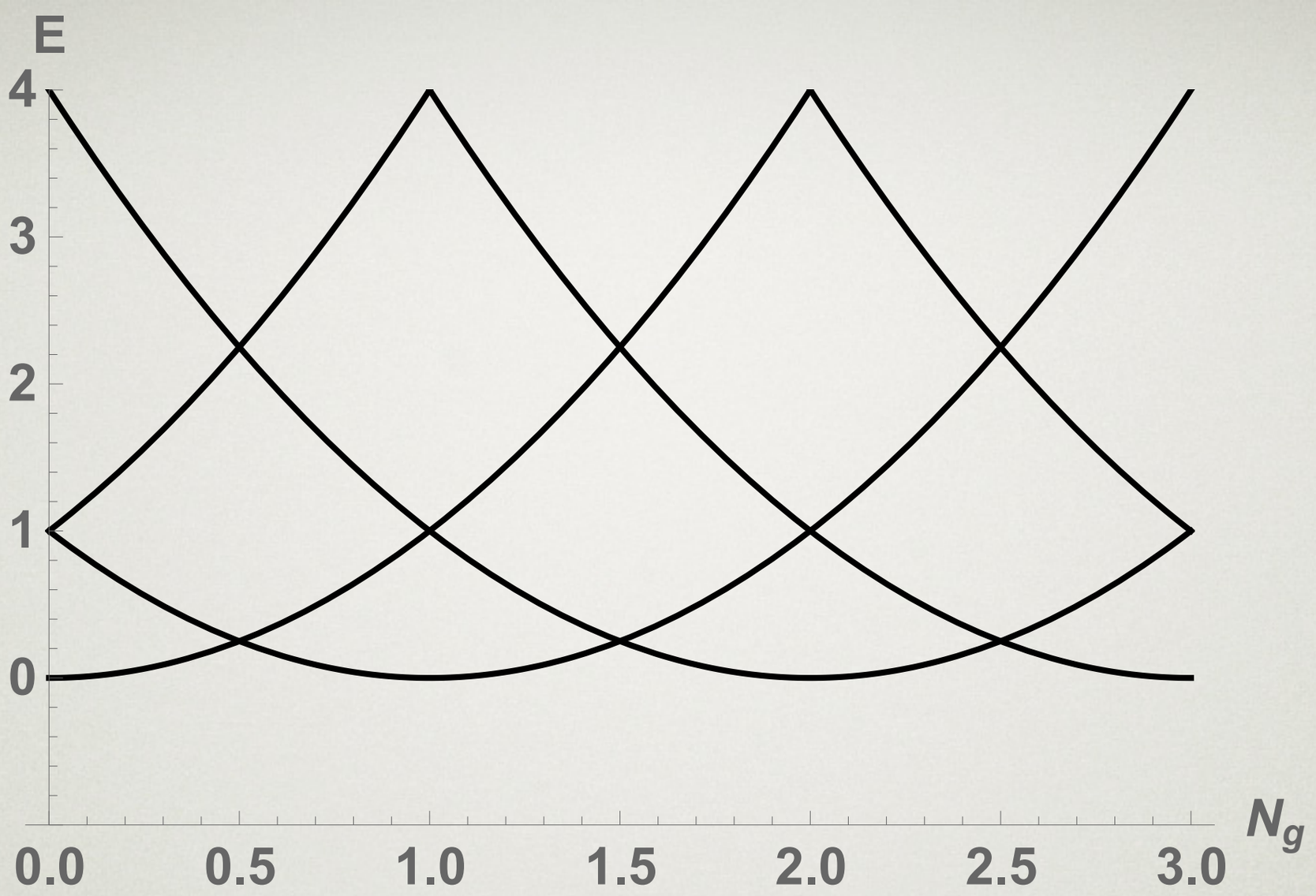
$$G = \left.\frac{\mathrm{d}I}{\mathrm{d}V}\right|_{V=0}$$

Coulomb blockade



$$E(N, V_g) = UN^2 - \alpha e V_g N = U(N - N_g)^2 + \text{konst.}$$

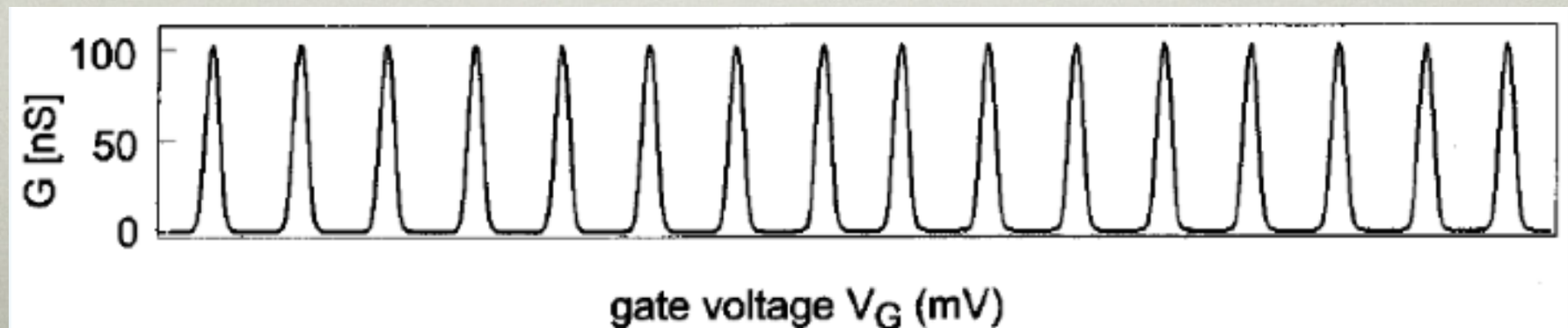
$$N_g = \frac{\alpha e V_g}{2U}$$



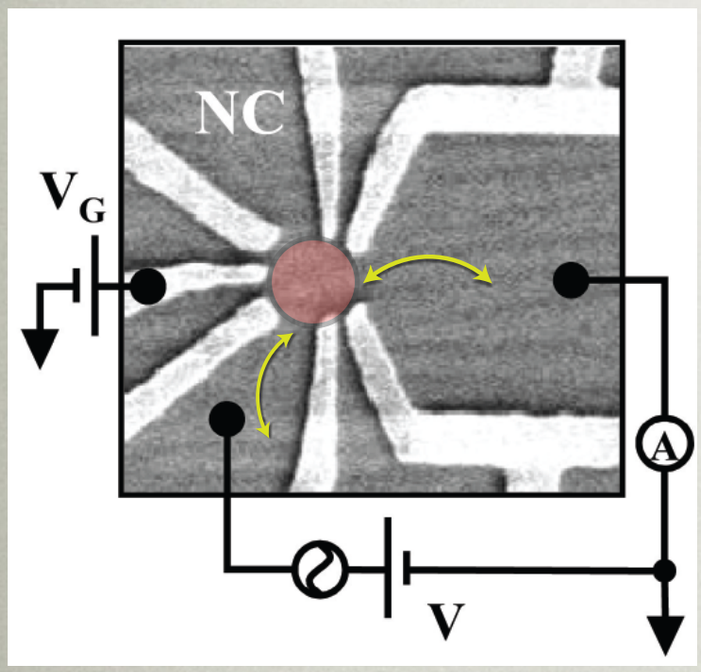
charge degeneracy points:

$$E(N+1, V_g) = E(N, V_g)$$

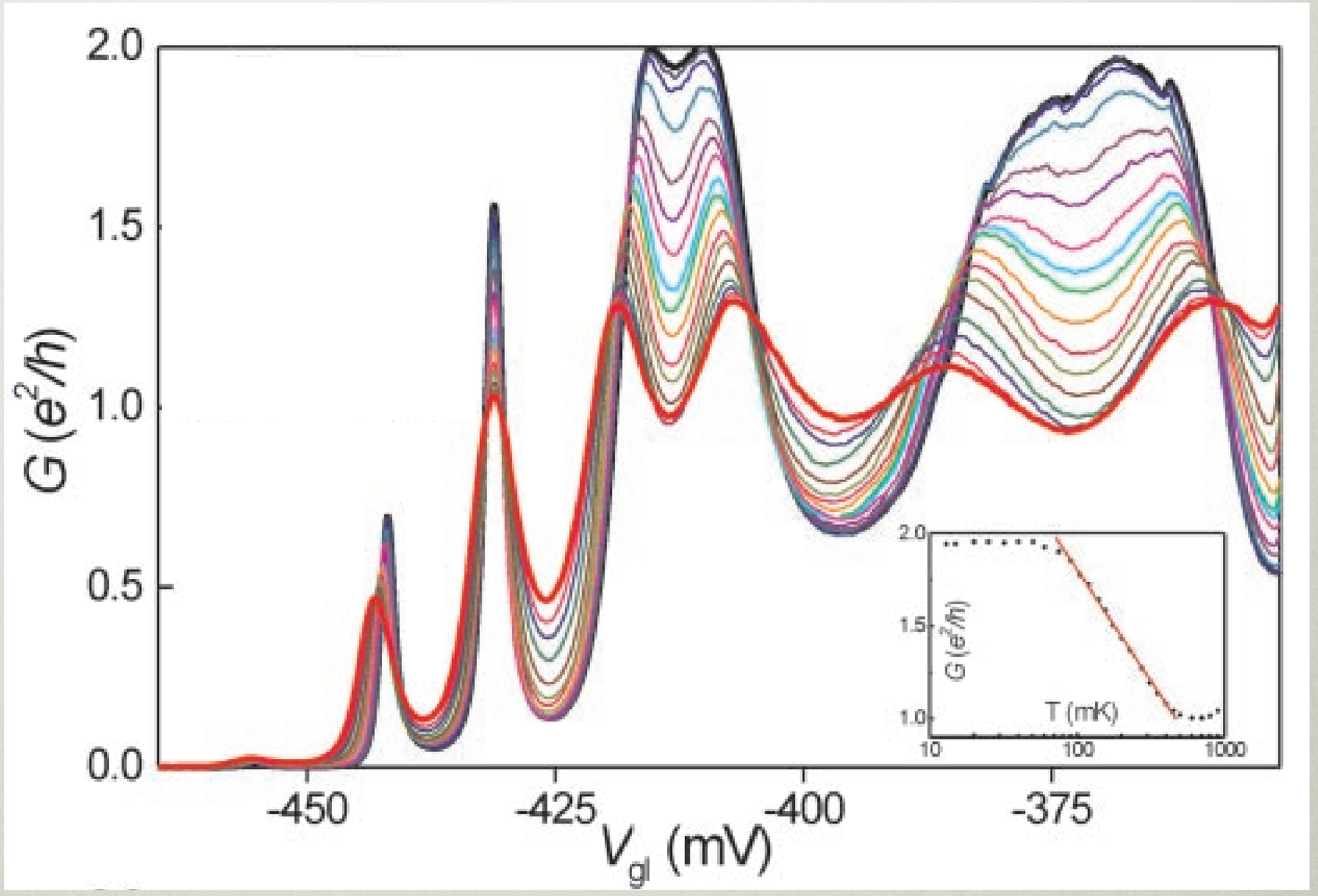
$$N_g = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$$



Kondo effect in quantum dots



Grobis et al., PRL 100, 246601 (2008)



$$G = \left.\frac{\mathrm{d}I}{\mathrm{d}V}\right|_{V=0}$$

unitary conductance: $G=G_0=2e^2/h$

W. G. van der Wiel et al., Science **289**, 2105 (2000)

Single-Impurity Anderson model

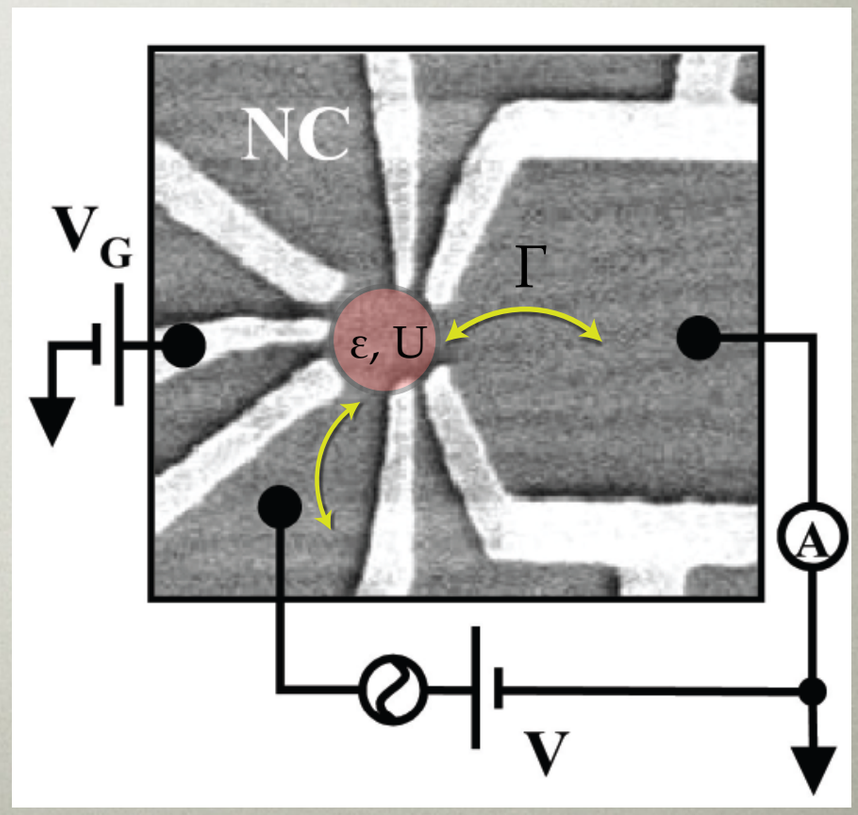
$$H = H_{\text{imp}} + H_{\text{band}} + H_{\text{hyb}}$$

$$H_{\text{imp}} = \sum_{\sigma} \epsilon\, n_\sigma + U\, n_\uparrow n_\downarrow \qquad n_\sigma = d_\sigma^\dagger d_\sigma$$

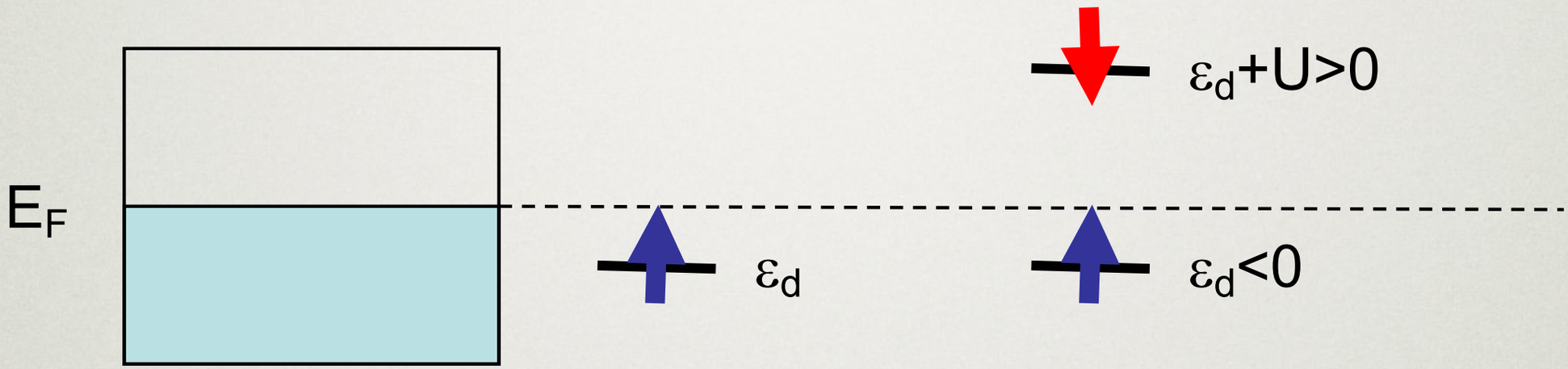
$$H_{\text{band}} = \sum_{\mathbf{k},\sigma} \epsilon_k\, c_{\mathbf{k},\sigma}^\dagger c_{\mathbf{k},\sigma}$$

$$H_{\text{hyb}} = \sum_{k,\sigma} \left(V_k\, c_{k,\sigma}^\dagger d_\sigma + \text{H.c.} \right)$$

$$\Delta(\omega) = \sum_k \frac{|V_k|^2}{\omega - \epsilon_k} \approx i\Gamma$$



Emergence of local moment in SIAM



Localized Magnetic States in Metals

P. W. Anderson
Phys. Rev. **124**, 41 – Published 1 October 1961

Article | References | Citing Articles (3,556) | PDF | Export Citation

Self-Consistent Equations Including Exchange and Correlation Effects

W. Kohn and L. J. Sham
Phys. Rev. **140**, A1133 – Published 15 November 1965

Physics See Focus story: Nobel Focus: Chemistry by Computer

Article | References | Citing Articles (24,384) | PDF | Export Citation

Special points for Brillouin-zone integrations

Hendrik J. Monkhorst and James D. Pack
Phys. Rev. B **13**, 5188 – Published 15 June 1976

Article | References | Citing Articles (21,250) | PDF | Export Citation

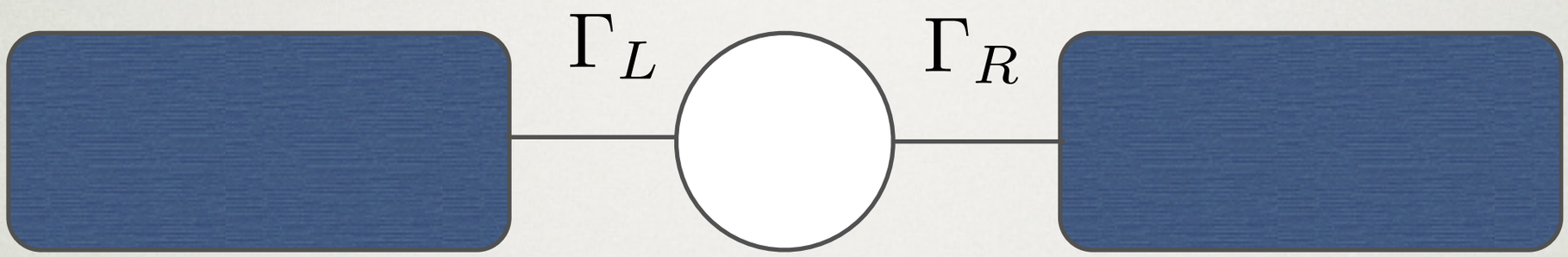
Inhomogeneous Electron Gas

P. Hohenberg and W. Kohn
Phys. Rev. **136**, B864 – Published 9 November 1964

Physics See Focus story: Nobel Focus: Chemistry by Computer

Article | References | Citing Articles (19,015) | PDF | Export Citation

Meir-Wingreen formula



$$I = \mathrm{d}/\mathrm{dt}\, N_L = i[H, N_L]$$

$$I = \frac{ie}{2h}\int d\epsilon \left(\mathrm{Tr}\left\{\left[f_L(\epsilon)\mathbf{\Gamma}^L - f_R(\epsilon)\mathbf{\Gamma}^R\right]\left(\mathbf{G}^r - \mathbf{G}^a\right)\right\} + \mathrm{Tr}\left\{\left(\mathbf{\Gamma}^L - \mathbf{\Gamma}^R\right)\mathbf{G}^<\right\}\right)$$

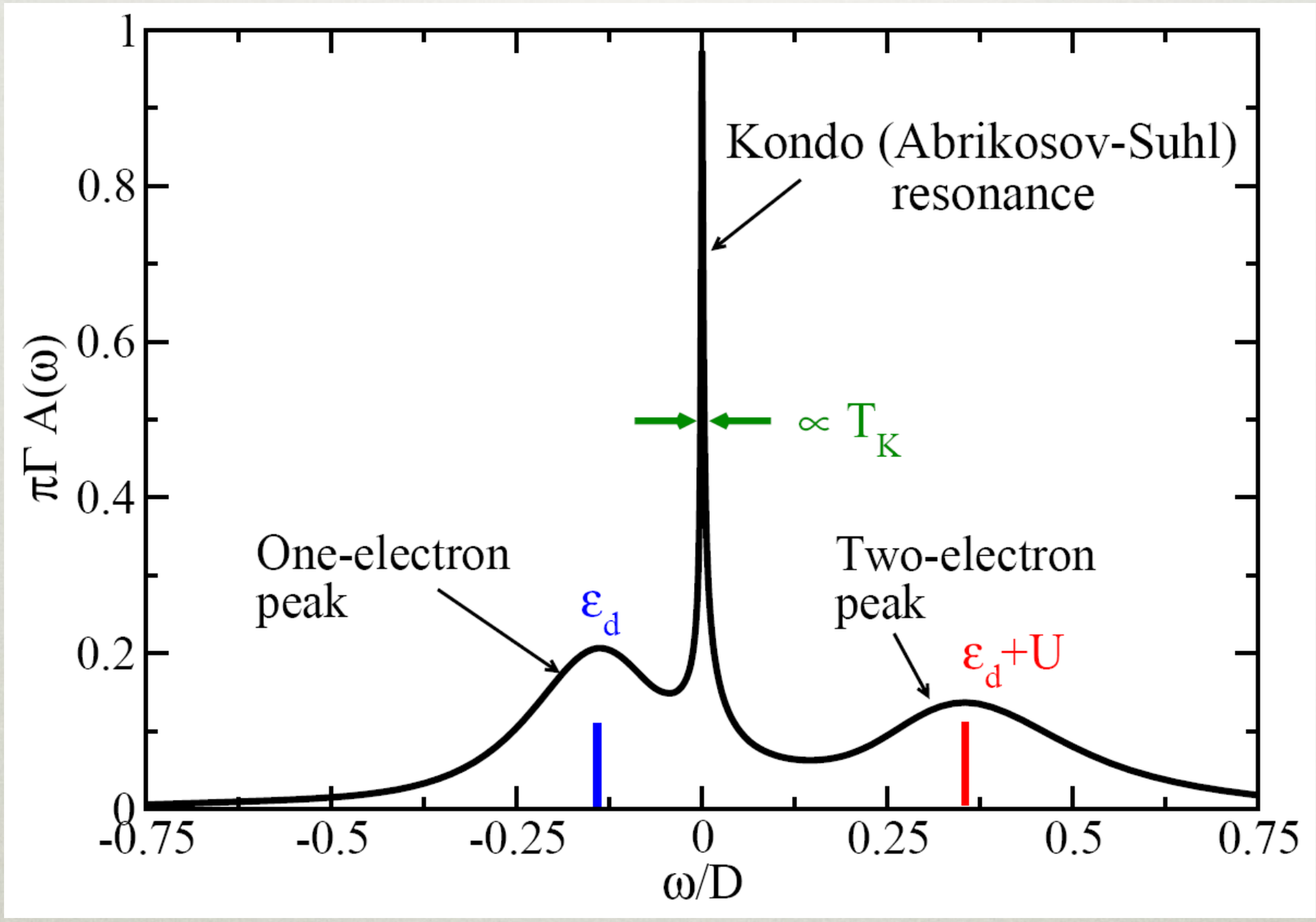
$$G(T) = G_0 \pi \Gamma \int_{-\infty}^{\infty} \mathrm{d}\omega \left(-\frac{\partial f}{\partial \omega}\right) A(\omega, T)$$

$$G = G_0 \pi \Gamma A(\omega = 0)$$

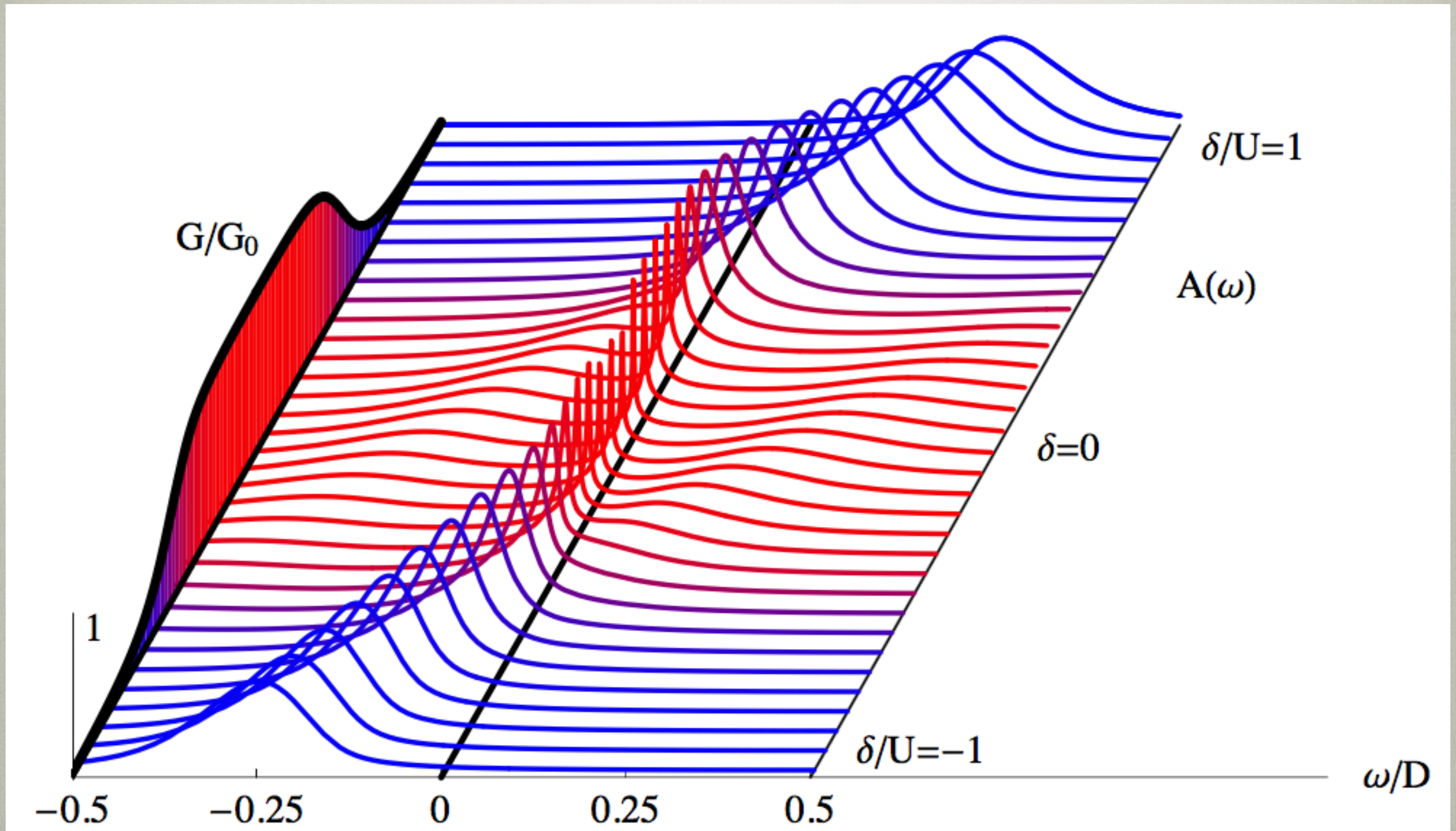
$$f(\omega) = \frac{1}{1 + e^{\beta\omega}}$$

Meir, Wingreen PRL 1992

Kondo resonance

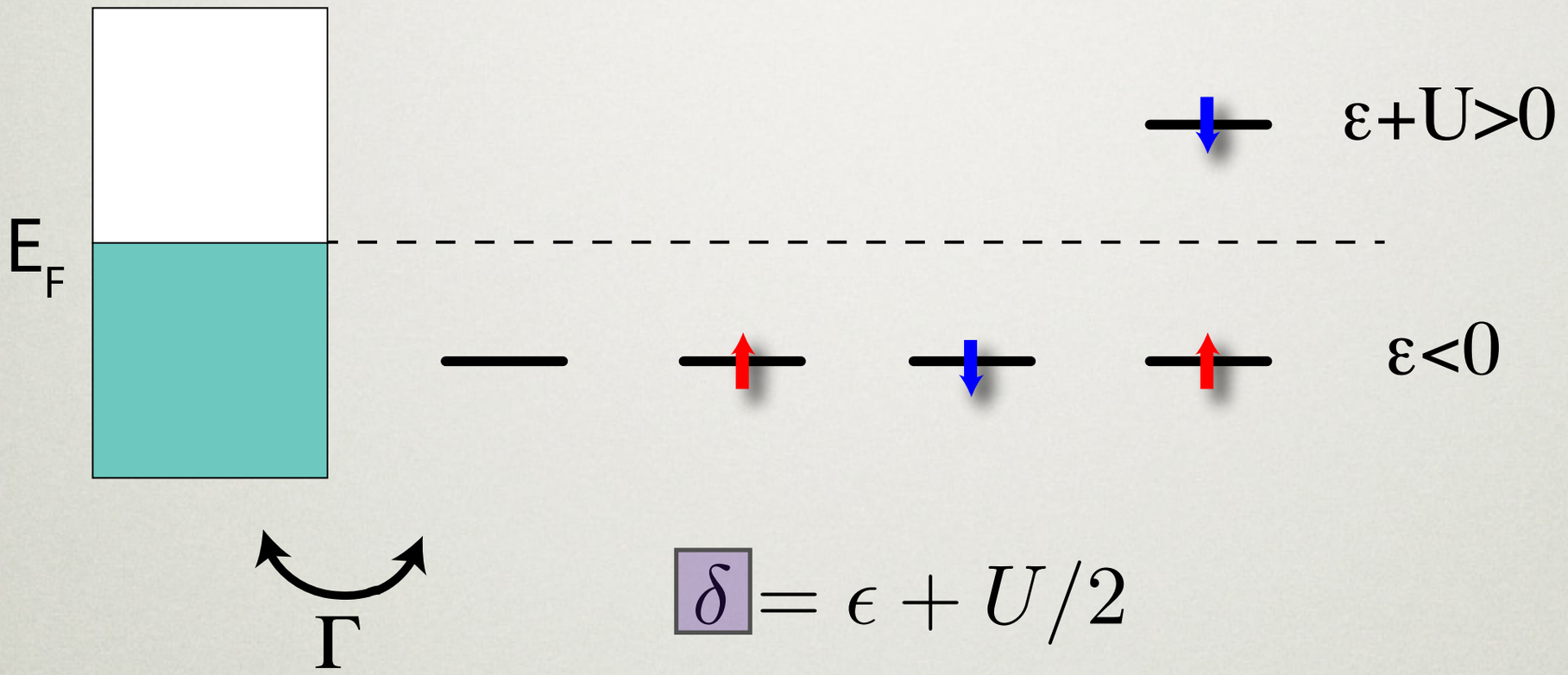


Frota, Oliveira, PRB 33, 7871 (1986), Costi, Hewson, Zlatić, JPCM 6, 2519 (1994)



$$\delta = \epsilon_d + U/2$$

Schrieffer-Wolff transformation



$$\delta = \epsilon + U/2$$

δ=0 is a special "particle-hole symmetric point" with ⟨n⟩=1

$$\rho J = \frac{8\Gamma}{\pi U}$$

THE FAMILY OF KONDO IMPURITY MODELS

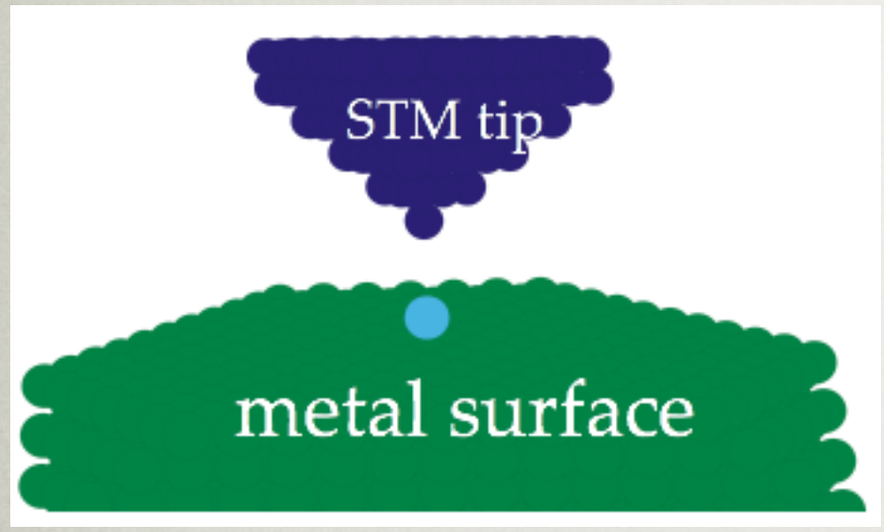
$$H = \sum_{\mathbf{k},\sigma,i} \epsilon_k c^\dagger_{\mathbf{k},\sigma,i} c_{\mathbf{k},\sigma,i} + \sum_i J \mathbf{s}_i \cdot \mathbf{S} + \mathbf{B} \cdot \mathbf{S} \qquad i=1,\ldots,N_{channels}$$

Classification according to 2S vs. $N_{channels}$

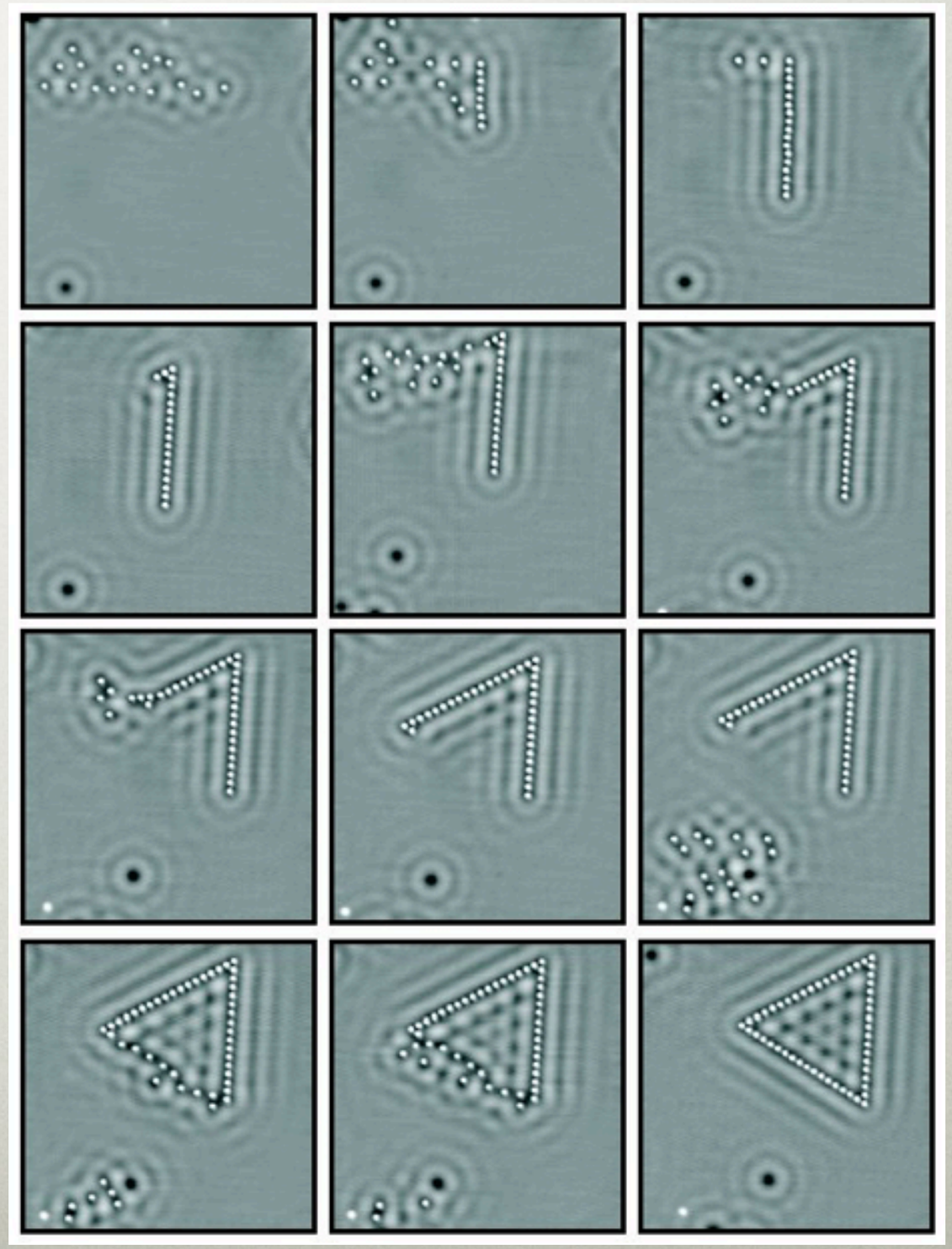
	fully screened Kondo model	underscreened Kondo model	overscreened Kondo model
impurity spin, S	1/2	1	1/2
$N_{channels}$	1	1	2
fixed point	Fermi liquid	singular Fermi liquid	non-Fermi liquid

P. Nozières and A. Blandin. J. Physique 41, 193 (1980)

Magnetic impurities on surfaces



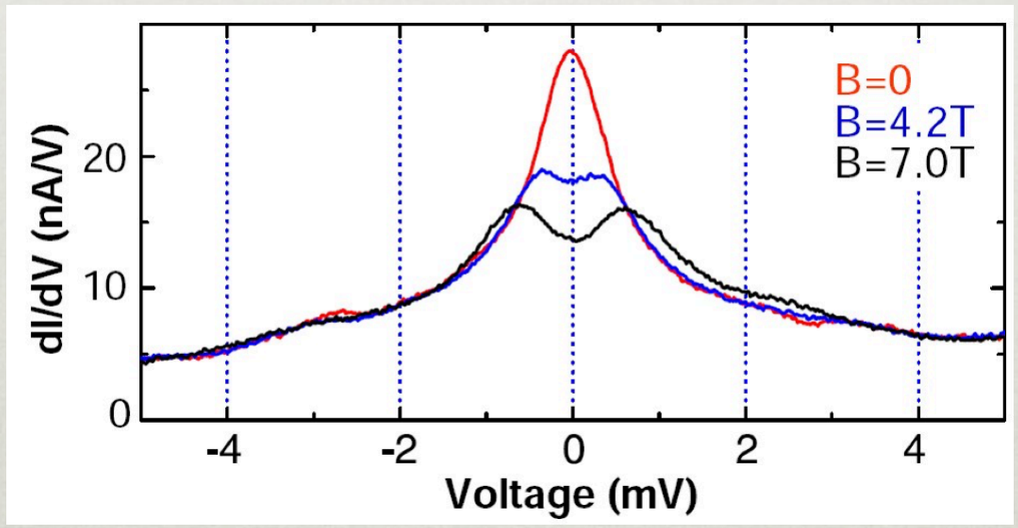




Kai Braun
FU Berlin

Tunneling spectroscopy

Bardeen tunneling formalism: $$\frac{dI(V)}{dV} \propto \rho_{\mathrm{local}}(eV)$$

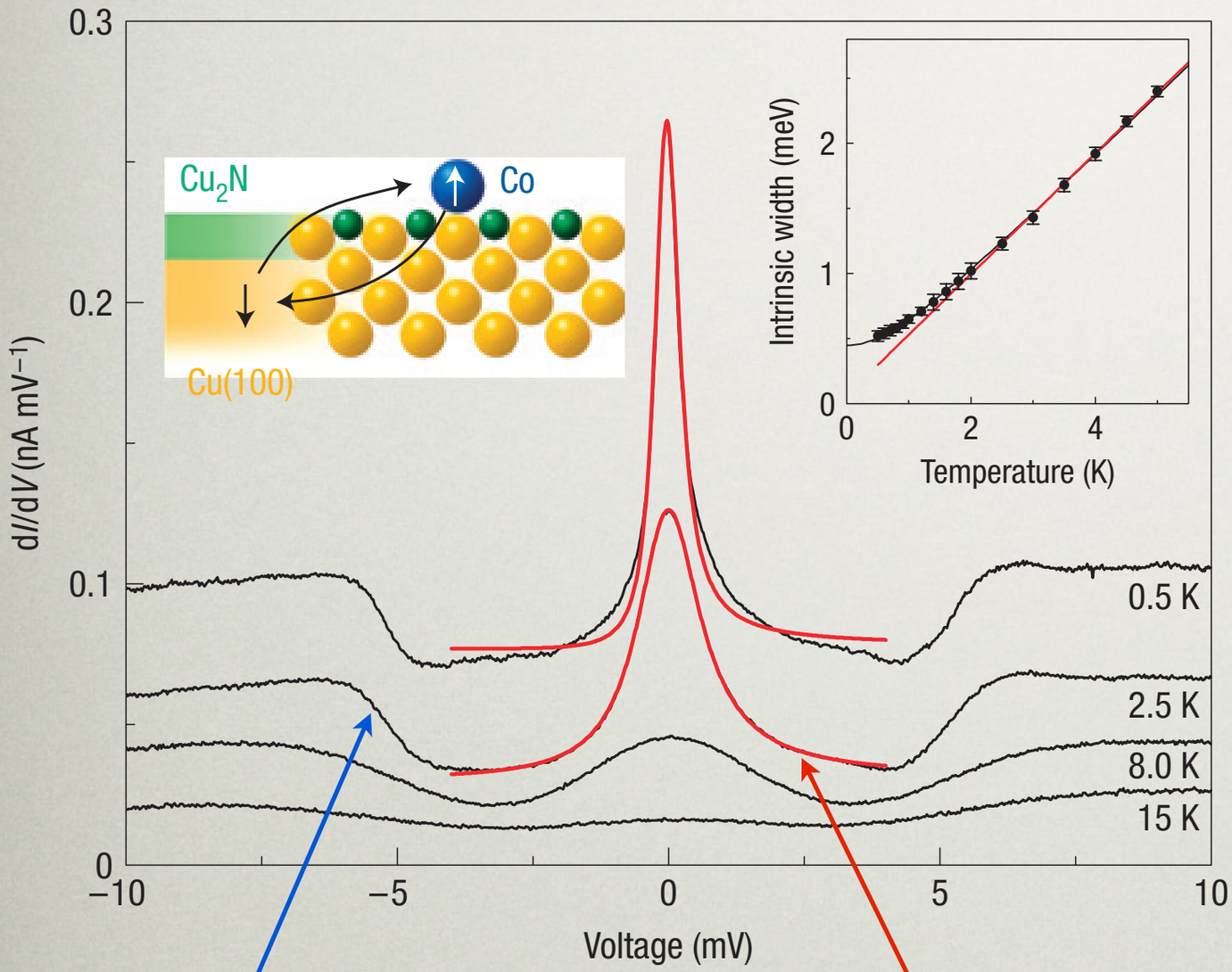


Heinrich et al., Science 2004

$Ti/Cu_2N/Cu(100)$

S=1/2

Magnetic adsorbates on surfaces of normal metals



S=3/2

$$H_{\text{aniso}} = DS_z^2 + E(S_x^2 - S_y^2)$$

$$D > 0$$

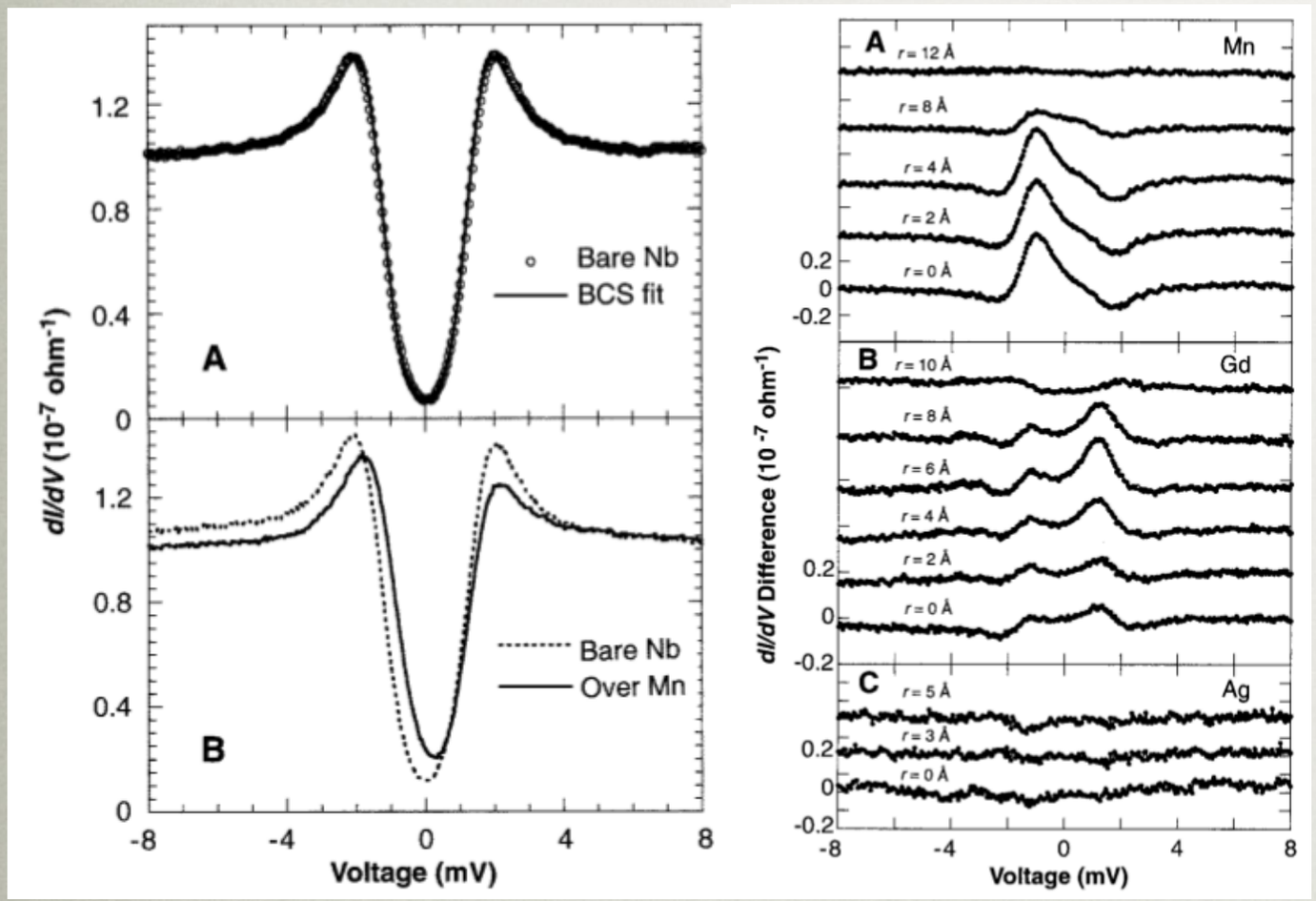
planar anisotropy

Kondo screening of $|S_z|=1/2$ doublet

inelastic $|S_z|=1/2 \rightarrow |S_z|=3/2$ transitions

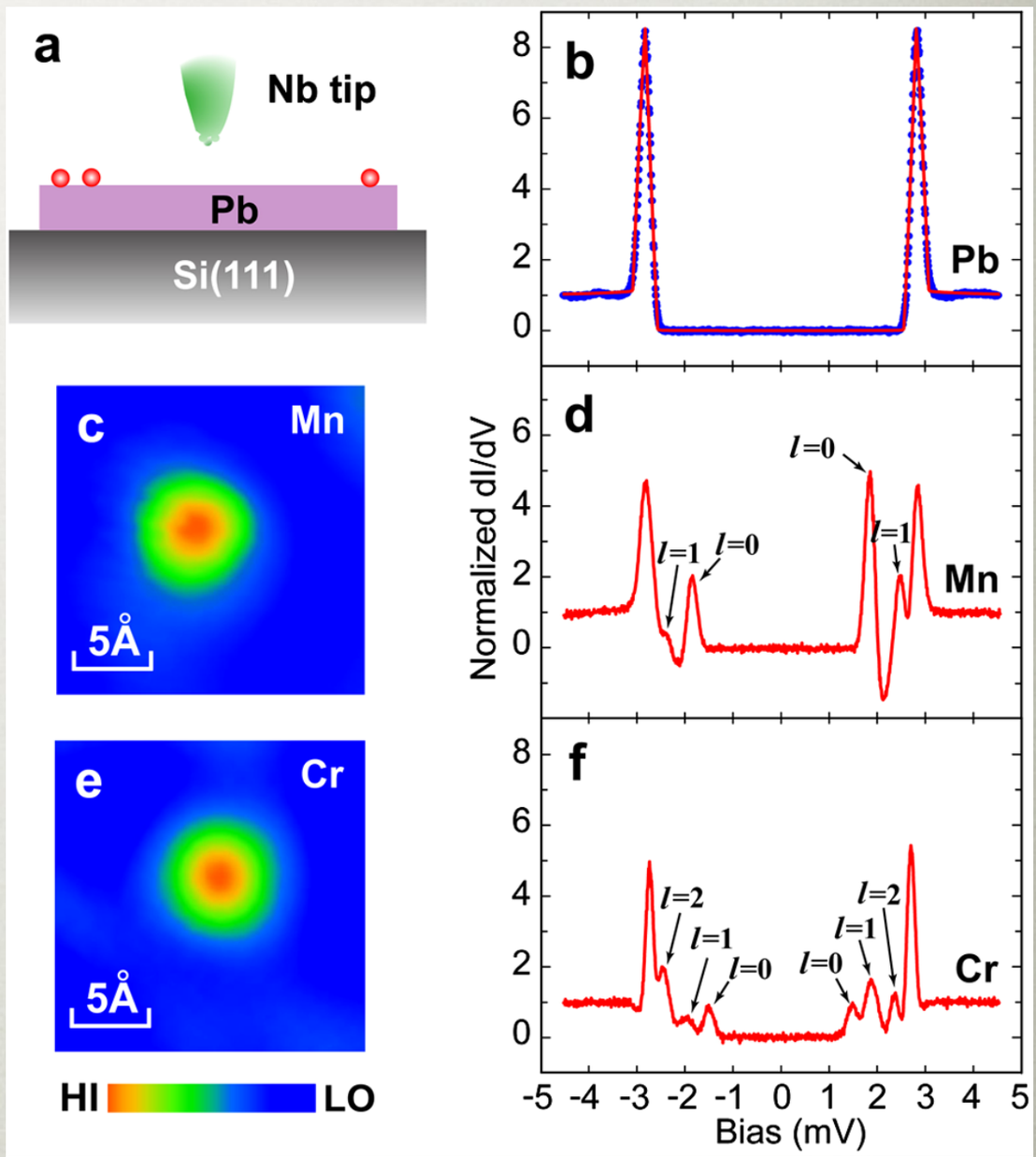
A. Otte et al., Nature Physics **4**, 847 (2008)

Magnetic adsorbates on surfaces of Superconductors



T=3.8K T_c(Nb)~9.2K

Yazdani, Jones, Lutz, Crommie, Eigler,
Science **275**, 1767 (1997)



T=400mK T_c(Pb)~7.2K

Shuai-Hua Ji et al.,
Phys. Rev. Lett. **100**, 226801 (2008)

Structure of the Hamiltonian:

- interacting localized (point-like) part
- non-interacting continuum part
- coupling term

Reduction to a 1D model

Because the impurity is point-like, the Hamiltonian maps to a 1D model.

k-space basis:

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}} c_{\mathbf{k}} + \sum_{\mathbf{k}} \left(V_{\mathbf{k}} c^\dagger_{\mathbf{k}} d + \text{H.c.} \right) + H_{\text{imp}}$$

Lanczos algorithm

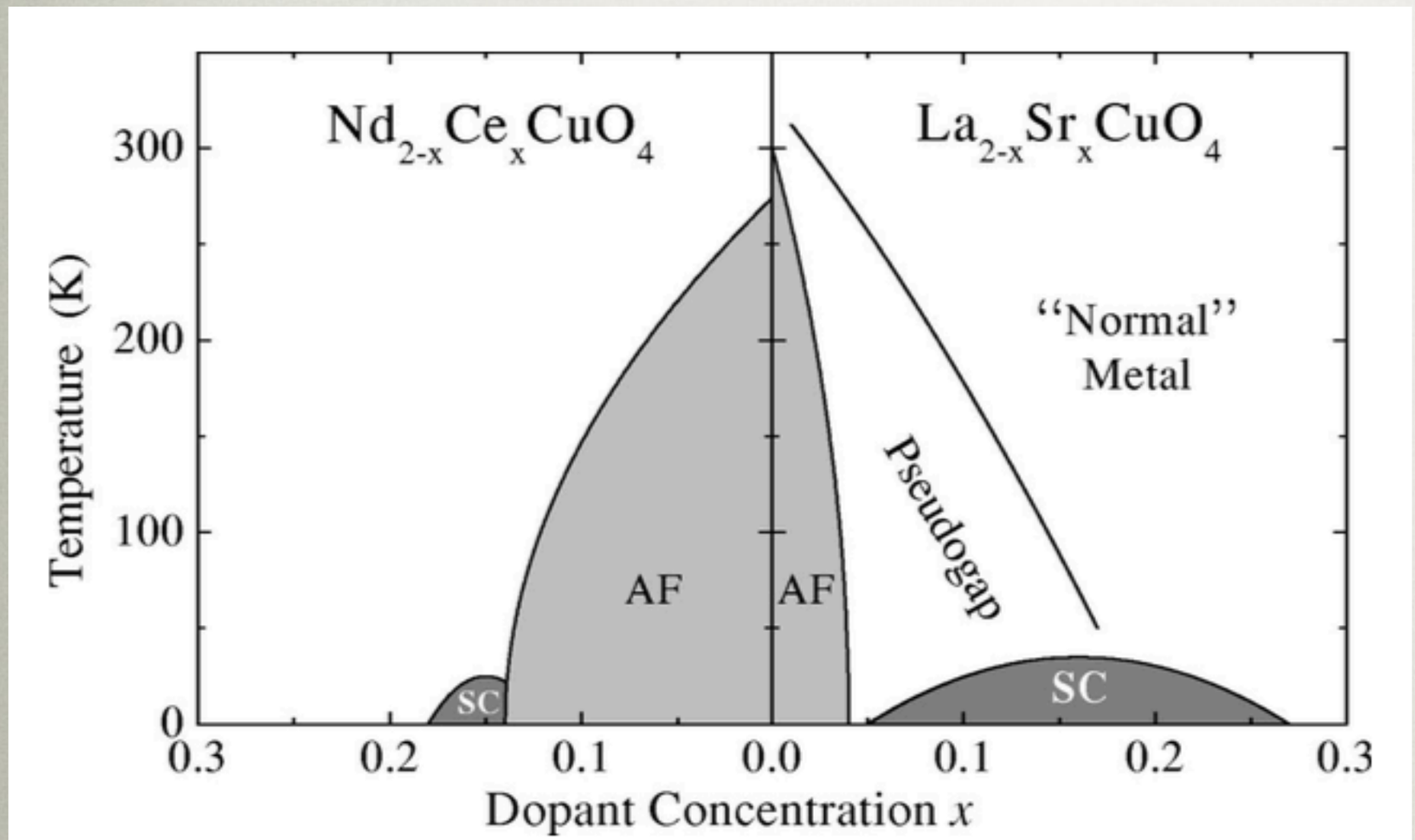
chain basis:

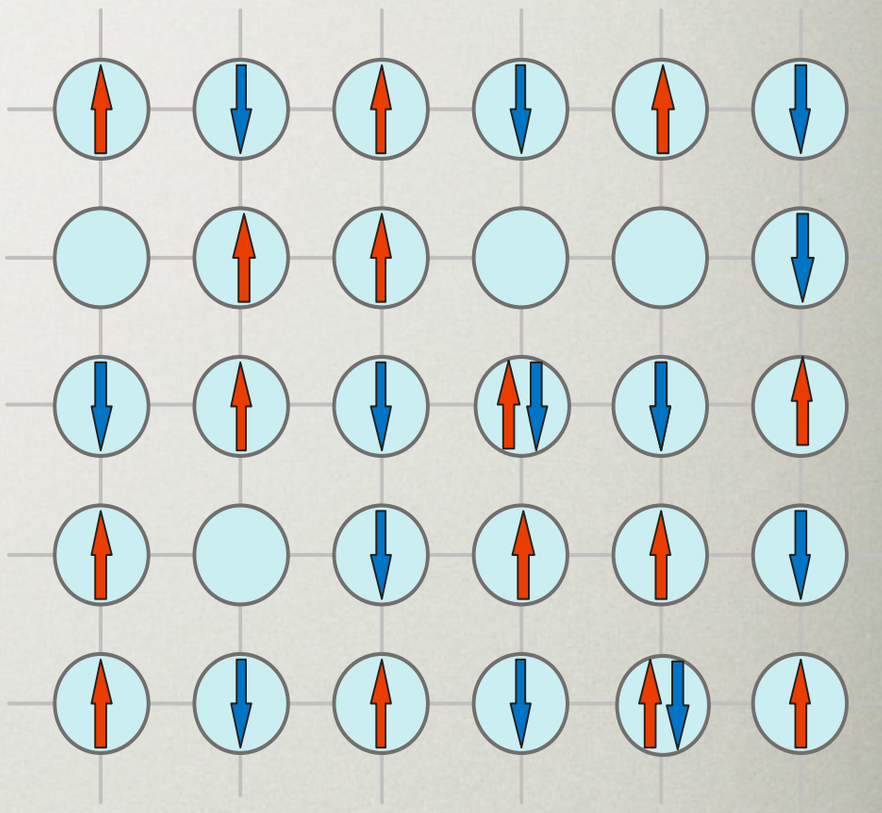
$$H = V(f_0^\dagger d + \text{H.c.}) + \sum_{n \geq 0} \left[\epsilon_n f_n^\dagger f_n + t_n \left(f_n^\dagger f_{n+1} + \text{H.c.} \right) \right] + H_{\text{imp}}$$

noninteracting 1D chain with interacting impurity attached to the edge

starting point for conformal field theory (CFT) approach

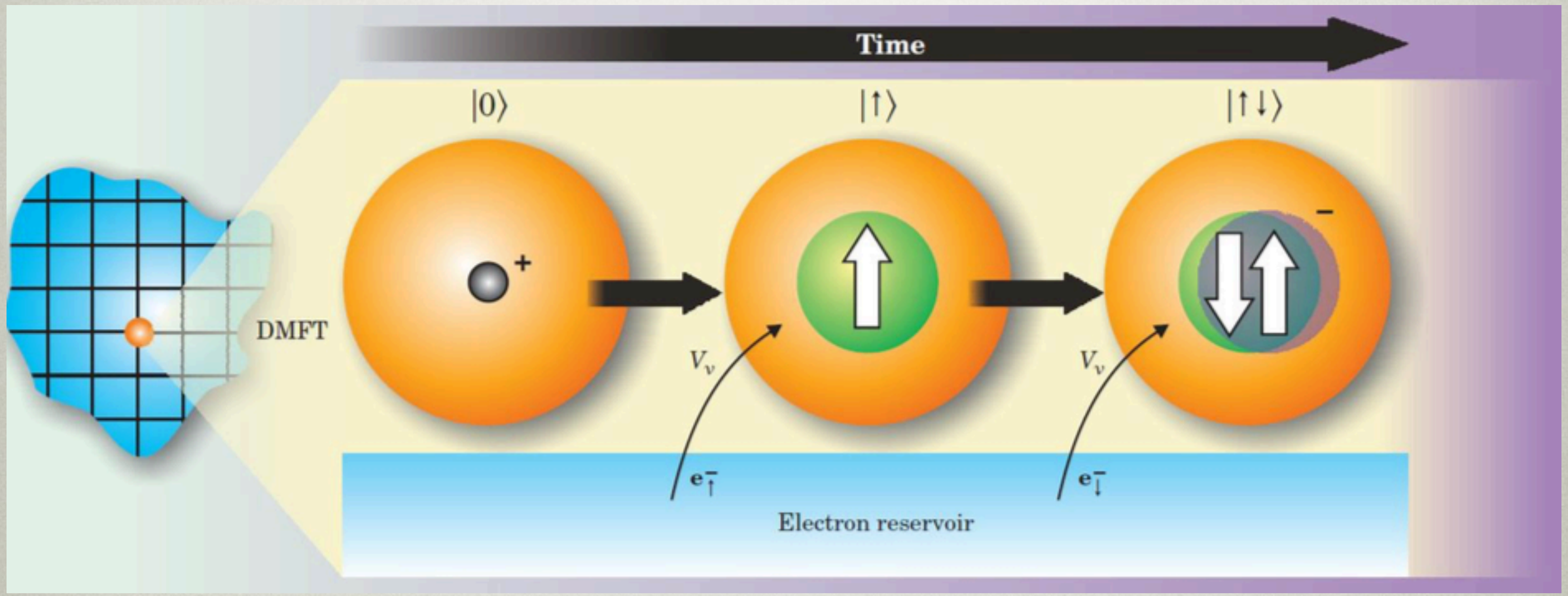
Strongly-correlated electron systems





$$H = -t \sum_{\langle i,j \rangle,\sigma} \left(c^\dagger_{i\sigma} c_{j\sigma} + \text{H.c.} \right) + U \sum_i n_{i\uparrow} n_{i\downarrow}$$

Hubbard model



Kotliar, Vollhardt, Physics Today (2004)

Dynamical mean field theory (DMFT)

1) local dynamics: exact calculation!
2) non-local correlations: static mean-field approximation

assumption:

$$\Sigma(k,\omega) \to \Sigma(\omega) \qquad \text{or} \qquad \Sigma_{i,j}(\omega) \to \delta_{ij}\Sigma(\omega)$$

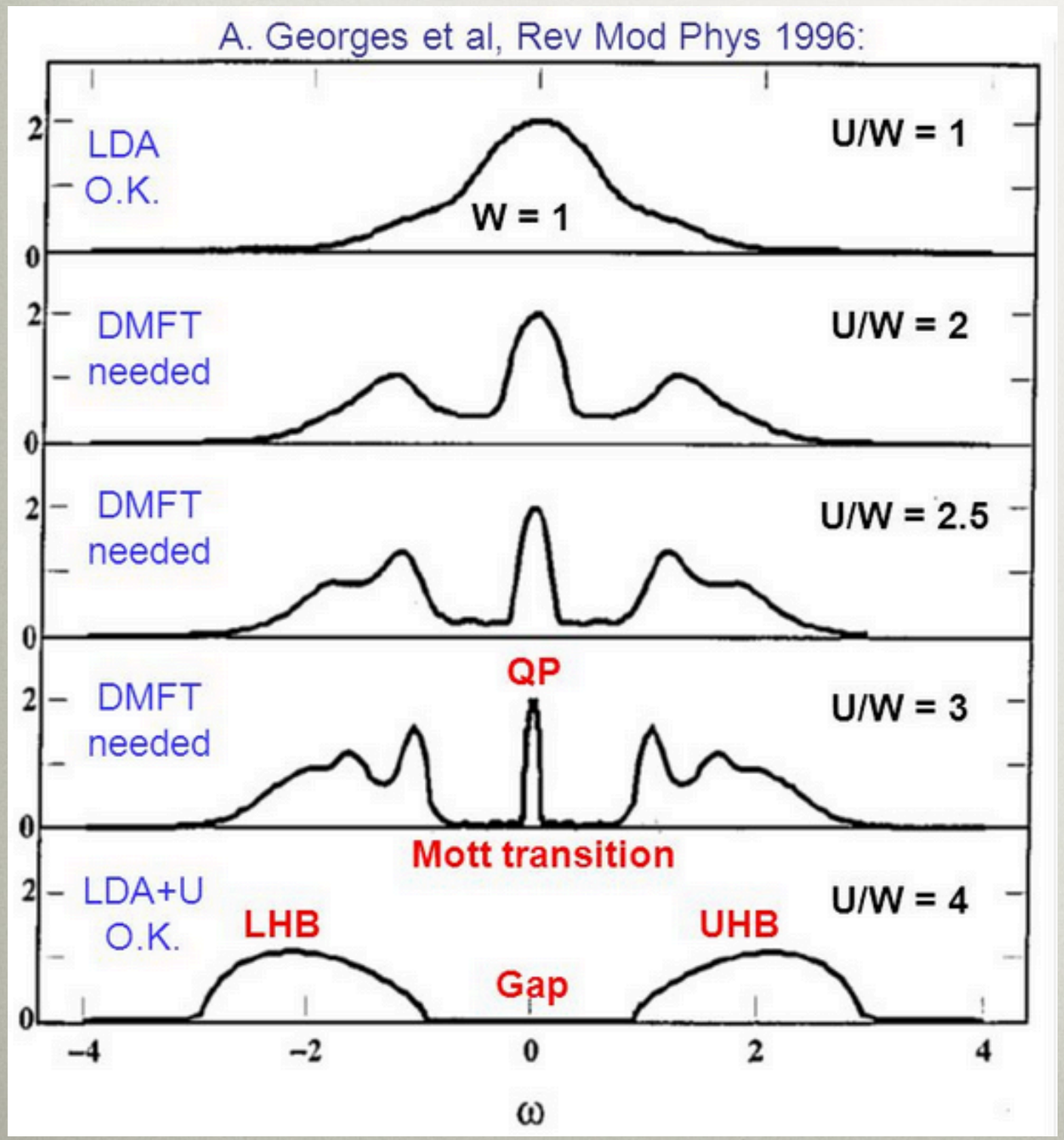
$$G_{\mathbf{k}}(z) = \frac{1}{z + \mu - \epsilon_{\mathbf{k}} - \Sigma(z)}$$

$$G_{\rm loc}(z) = \frac{1}{N}\sum_{\mathbf{k}} \frac{1}{z + \mu - \epsilon_{\mathbf{k}} - \Sigma(z)} = \int \frac{\rho_0(\epsilon)\,{\rm d}\epsilon}{[z + \mu - \Sigma(z)] - \epsilon} = G_0[z + \mu - \Sigma(z)]$$

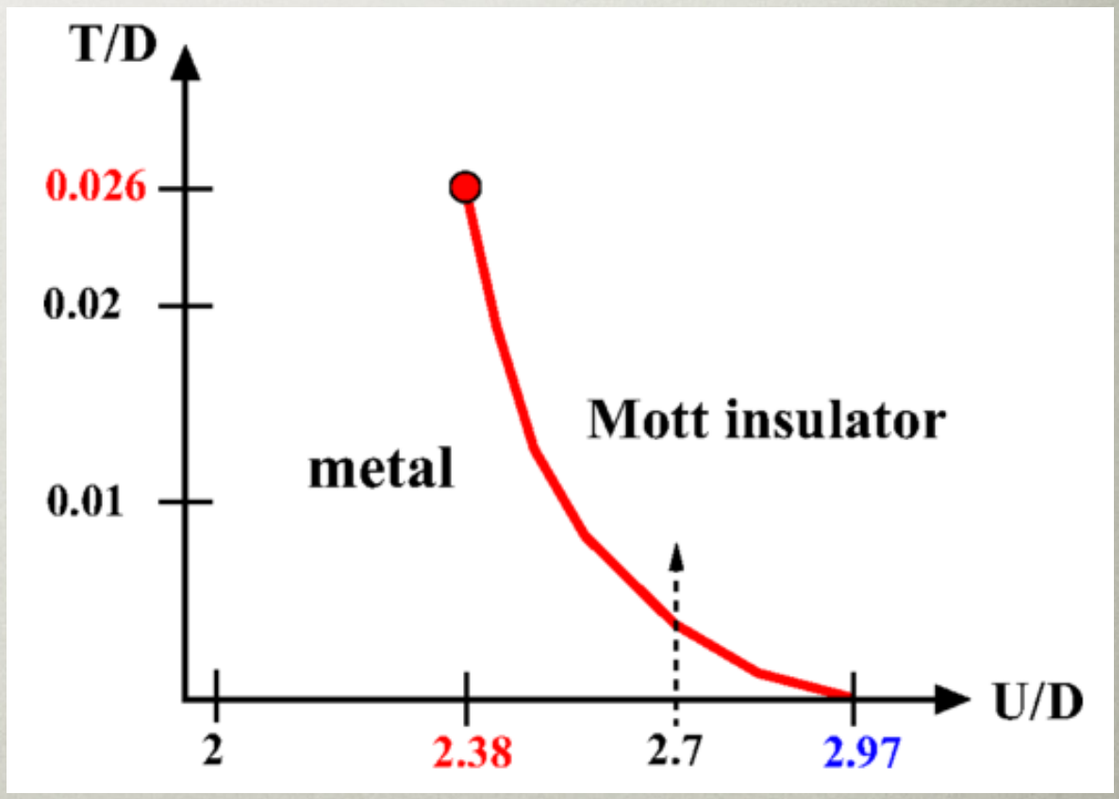
$$\Delta(z) = z + \mu - \left[G_{\rm loc}^{-1}(z) + \Sigma(z)\right]$$

A. Georges et al., Rev. Mod. Phys. (1996)

Interaction-driven Mott transition



U: interaction strength
W: bandwidth (kinetic energy)



Green's functions - review

$$G_{AB}(t) = \langle\langle A; B\rangle\rangle_t := -i\theta(t)\langle[A(t), B(0)]_\pm\rangle$$

+ if A and B are fermionic operators
- if A and B are bosonic operators

canonical:

$$\langle\hat{O}\rangle = \mathrm{Tr}\left[\rho\hat{O}\right] \qquad \rho = \frac{e^{-\beta H}}{Z}$$

$$Z = \mathrm{Tr}[e^{-\beta H}]$$

grand canonical:

$$\rho = \frac{e^{-\beta K}}{\mathcal{Z}}$$

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta K}]$$

$$K = H - \mu N$$

Heisenberg representation: $A(t) = e^{iHt}Ae^{-iHt}$ $\qquad A(t) = e^{iKt}Ae^{-iKt}$

Green's function in frequency domain

Laplace transformation:

$$G_{AB}(z) = \langle\langle A; B\rangle\rangle_z = \int_0^\infty \mathrm{d}t e^{izt} \langle\langle A; B\rangle\rangle_t, \quad \Im z > 0$$

Inverse Laplace transformation:

$$G_{AB}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \mathrm{d}\omega e^{-i(\omega+i\delta)t} G_{AB}(\omega + i\delta)$$

Impurity Green's function (for SIAM):

$$G(z) = \langle\langle d; d^\dagger \rangle\rangle_z$$

Equations of motion

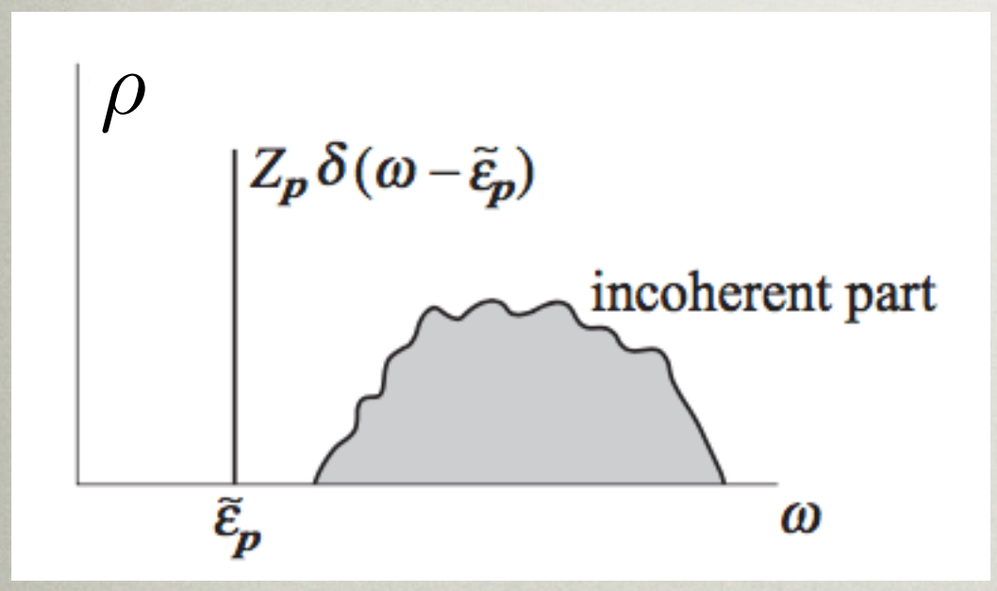
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\langle A;B\rangle\rangle_t = \langle\langle \dot{A};B\rangle\rangle_t - i\delta(t)\langle[A,B]_\pm\rangle$$

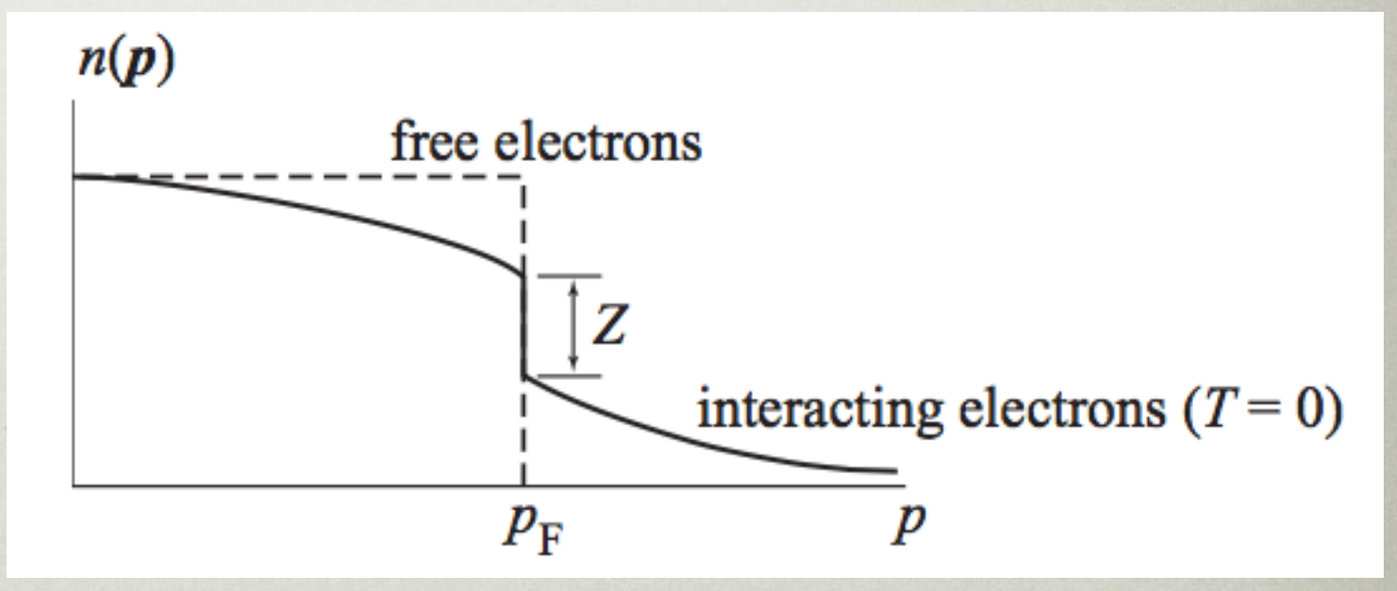
$$\begin{aligned} z\langle\langle A;B\rangle\rangle_z &= \langle[A,B]_\eta\rangle + \langle\langle[A,H]_\eta;B\rangle\rangle_z \\ &= \langle[A,B]_\eta\rangle + \langle\langle A;[H,B]_\eta\rangle\rangle_z \end{aligned}$$

example 1: free electrons

$$H = \sum_k \epsilon_k c_k^\dagger c_k \qquad \langle\langle c_{k'}, c_k^\dagger \rangle\rangle_z = \frac{\delta_{kk'}}{z - \epsilon_k} \qquad \rho_k(\omega) = \delta(\omega - \epsilon_k)$$

example 2: Fermi-liquid quasiparticles





quasiparticle weight $0 < Z \leq 1$

Anatomy of impurity Green's function

$$G(z) = \frac{1}{z - \epsilon - \Delta(z) - \Sigma(z)}$$

hybridisation function

self-energy function

Hybridisation function

$$\Delta(z) = \sum_k |V_k|^2 \frac{1}{z - \epsilon_k}$$

$$\Gamma(\omega) = -\mathrm{Im}\,\Delta(z + i0^+)$$

$$\Gamma(\omega) = \pi \sum_k |V_k|^2 \delta(\omega - \epsilon_k) \approx \pi \rho_0 |V_{k_F}|^2$$

Hybridization function fully describes the effect of the conduction band on the impurity.
This is possible because the band is non-interacting.

Plemelj formula

$$\frac{1}{x - i0^+} = P\frac{1}{x} + i\pi\delta(x)$$

Sokhotski–Plemelj theorem

From Wikipedia, the free encyclopedia

Not to be confused with Casorati–Sokhotski–Weierstrass theorem.

The **Sokhotski–Plemelj theorem** (Polish spelling is *Sochocki*) is a theorem in complex analysis, which helps in evaluating certain integrals. The real-line version of it (see below) is often used in physics, although rarely referred to by name. The theorem is named after Julian Sochocki, who proved it in 1868, and Josip Plemelj, who rediscovered it as a main ingredient of his solution of the Riemann-Hilbert problem in 1908.

Contents [hide]

1 Statement of the theorem
2 Version for the real line
3 Proof of the real version
4 Physics application
5 See also
6 References

Let C be a smooth closed simple curve in the plane, and φ an analytic function on C. Then the Cauchy-type integral

$$\frac{1}{2\pi i}\int_C \frac{\varphi(\zeta)\,d\zeta}{\zeta - z},$$

defines two analytic functions of z, ϕ_i inside C and ϕ_e outside.[*clarification needed*] The Sokhotski–Plemelj formulas relate the limiting boundary values of these two analytic functions at a point z on C and the Cauchy principal value $\mathcal{P}$ of the integral:

$$\phi_i(z) = \frac{1}{2\pi i}\mathcal{P}\int_C \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} + \frac{1}{2}\varphi(z),$$

$$\phi_e(z) = \frac{1}{2\pi i}\mathcal{P}\int_C \frac{\varphi(\zeta)\,d\zeta}{\zeta - z} - \frac{1}{2}\varphi(z).$$

Proof of the real version [edit]

A simple proof is as follows.

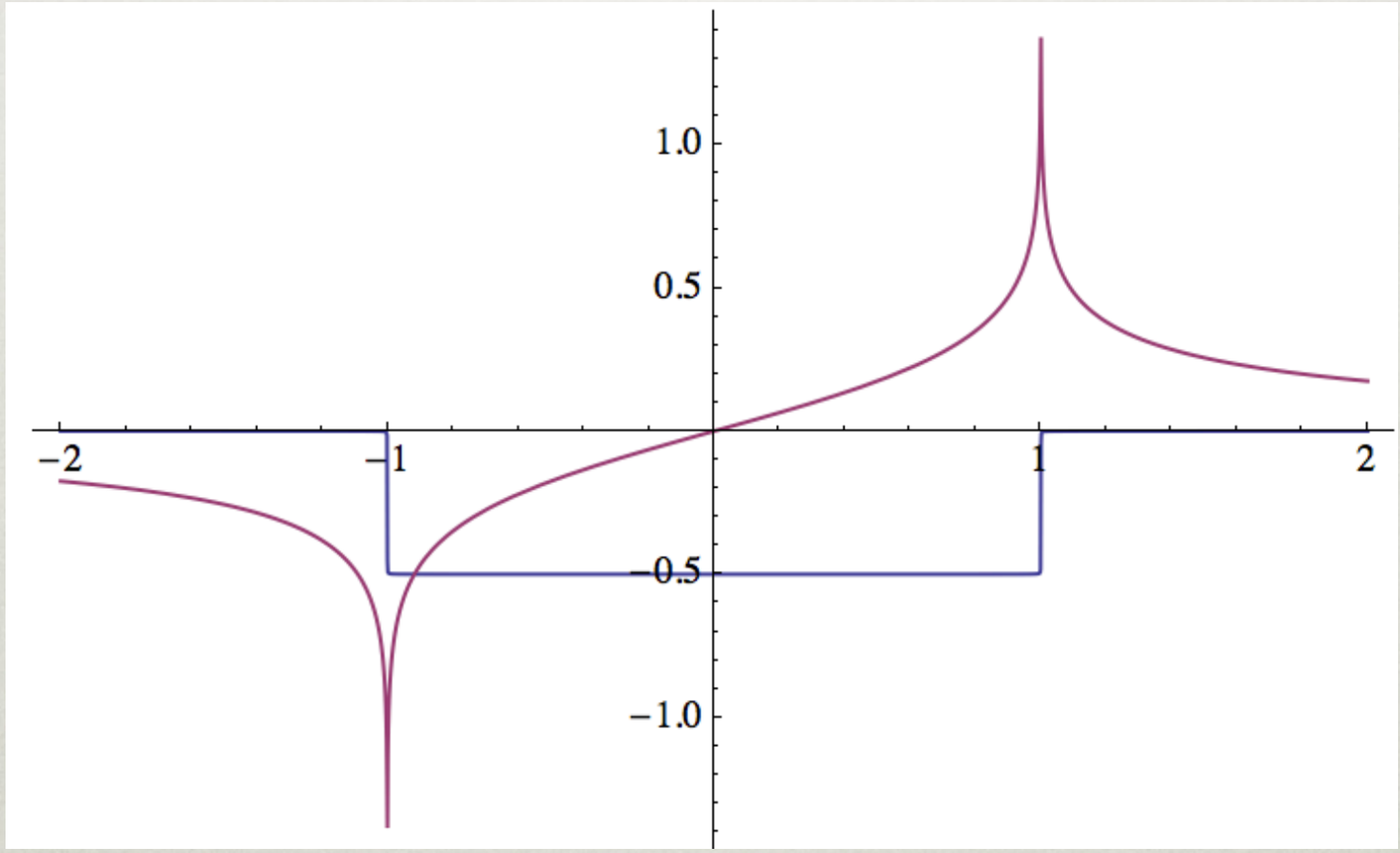
$$\lim_{\varepsilon\to 0^+}\int_a^b \frac{f(x)}{x\pm i\varepsilon}\,dx = \mp i\pi \lim_{\varepsilon\to 0^+}\int_a^b \frac{\varepsilon}{\pi(x^2+\varepsilon^2)}f(x)\,dx + \lim_{\varepsilon\to 0^+}\int_a^b \frac{x^2}{x^2+\varepsilon^2}\,\frac{f(x)}{x}\,dx.$$

For the first term, we note that $\varepsilon/\pi(x^2+\varepsilon^2)$ is a nascent delta function, and therefore approaches a Dirac delta function in the limit. Therefore, the first term equals $\mp i\pi\, f(0)$.

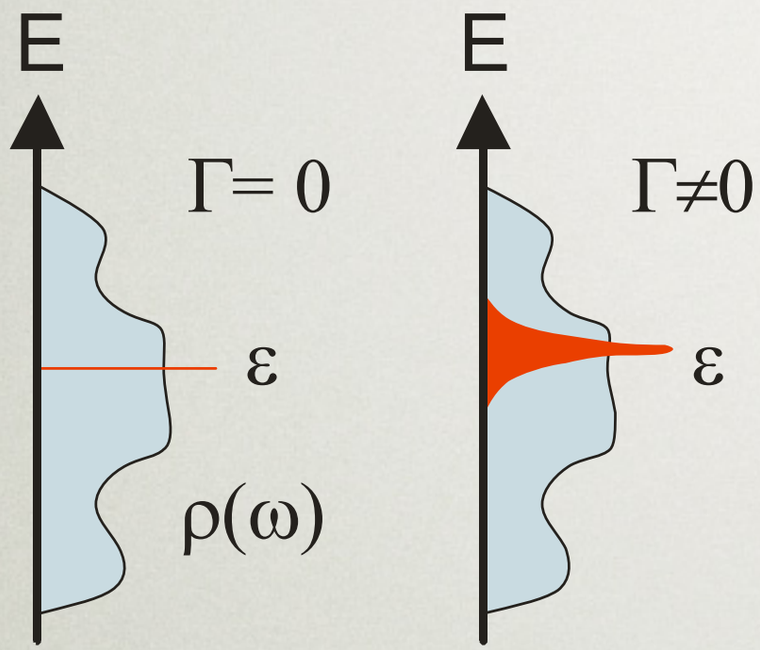
For the second term, we note that the factor $x^2/(x^2+\varepsilon^2)$ approaches 1 for $|x| \gg \varepsilon$, approaches 0 for $|x| \ll \varepsilon$, and is exactly symmetric about 0. Therefore, in the limit, it turns the integral into a Cauchy principal value integral.

HYBRIDISATION FUNCTION FOR FLAT BAND

$$\Delta(z) = \frac{\Gamma}{2\pi} \ln \frac{z+1}{z-1}$$



Resonant-level model



$$A(\omega) = -\frac{1}{\pi}\mathrm{Im}\left(\frac{1}{\omega - \epsilon + \Delta(\omega)}\right)$$

$$\Delta(\omega) = \Gamma\left[i + \frac{1}{\pi}\ln\left(\frac{1-\omega/D}{1+\omega/D}\right)\right]$$

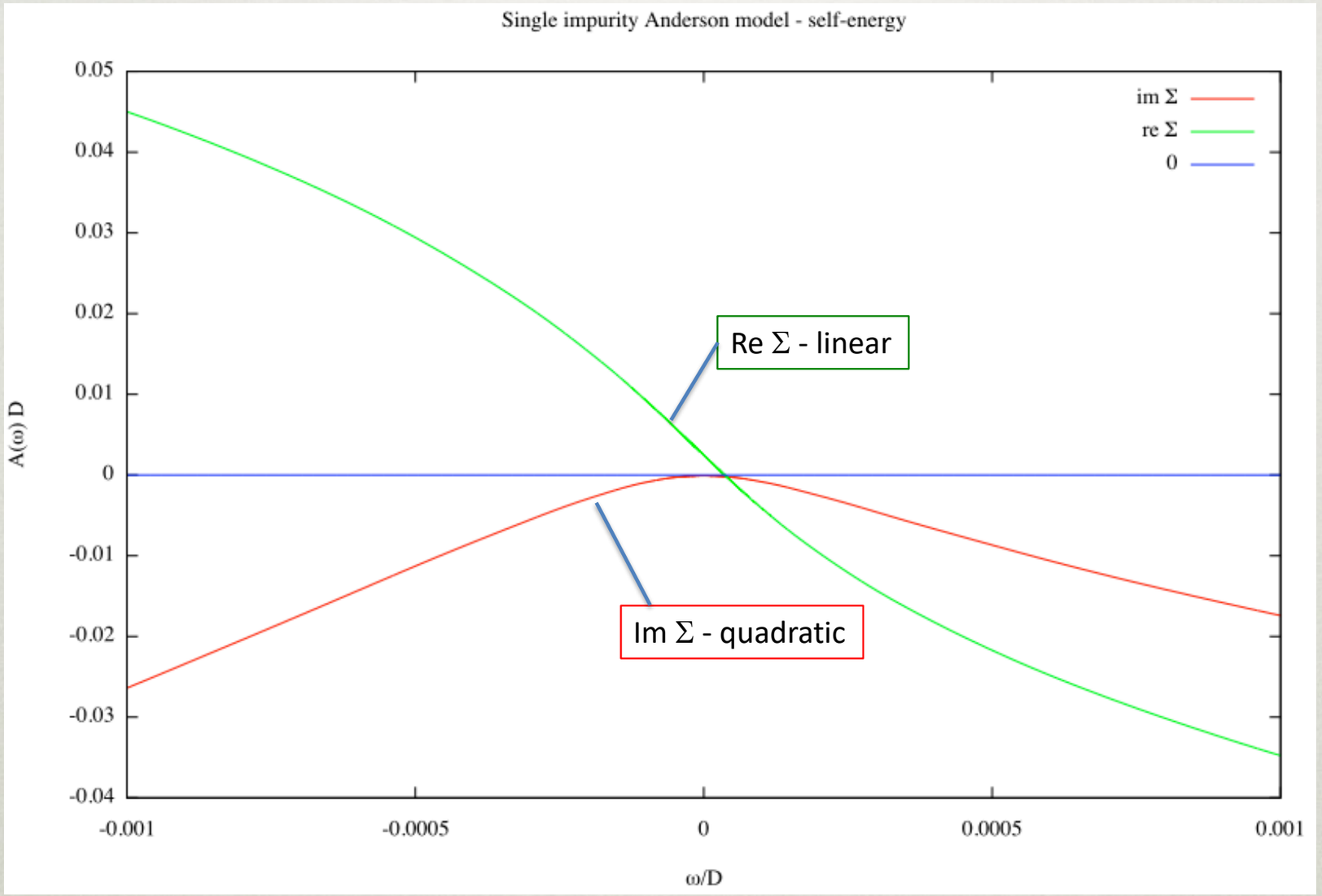
Self-energy

$$\Sigma(\omega) = \frac{\langle\langle [d, H_{\mathrm{imp}}]; d^\dagger \rangle\rangle}{\langle\langle d; d^\dagger \rangle\rangle}$$

SIAM: $$\Sigma_\sigma(\omega) = \frac{\langle\langle U n_{\bar{\sigma}} d_\sigma; d_\sigma^\dagger \rangle\rangle}{\langle\langle d_\sigma; d_\sigma^\dagger \rangle\rangle}$$

Self-energy function describes the effect of the local interaction on the impurity at the level of single-particle excitations.

Σ AT LOW FREQUENCIES



Characteristic behavior of Fermi-liquid systems.

$$\mathrm{Im}\,\Sigma(\omega, T) = a(\omega^2 + \pi^2 T^2) + b\omega^3 + c\omega T^2 + \ldots$$

Fermi-liquid form of G

$$G(z) = \frac{Z}{z - \epsilon} + G_{\text{incoh}}(z)$$

$$Z = \frac{1}{1 - \partial \text{Re}\, \Sigma(\omega)/\partial\omega|_{\omega=0}}$$

Physical interpretation of Σ

$$|\mathrm{Im}\,\Sigma(\omega)|$$

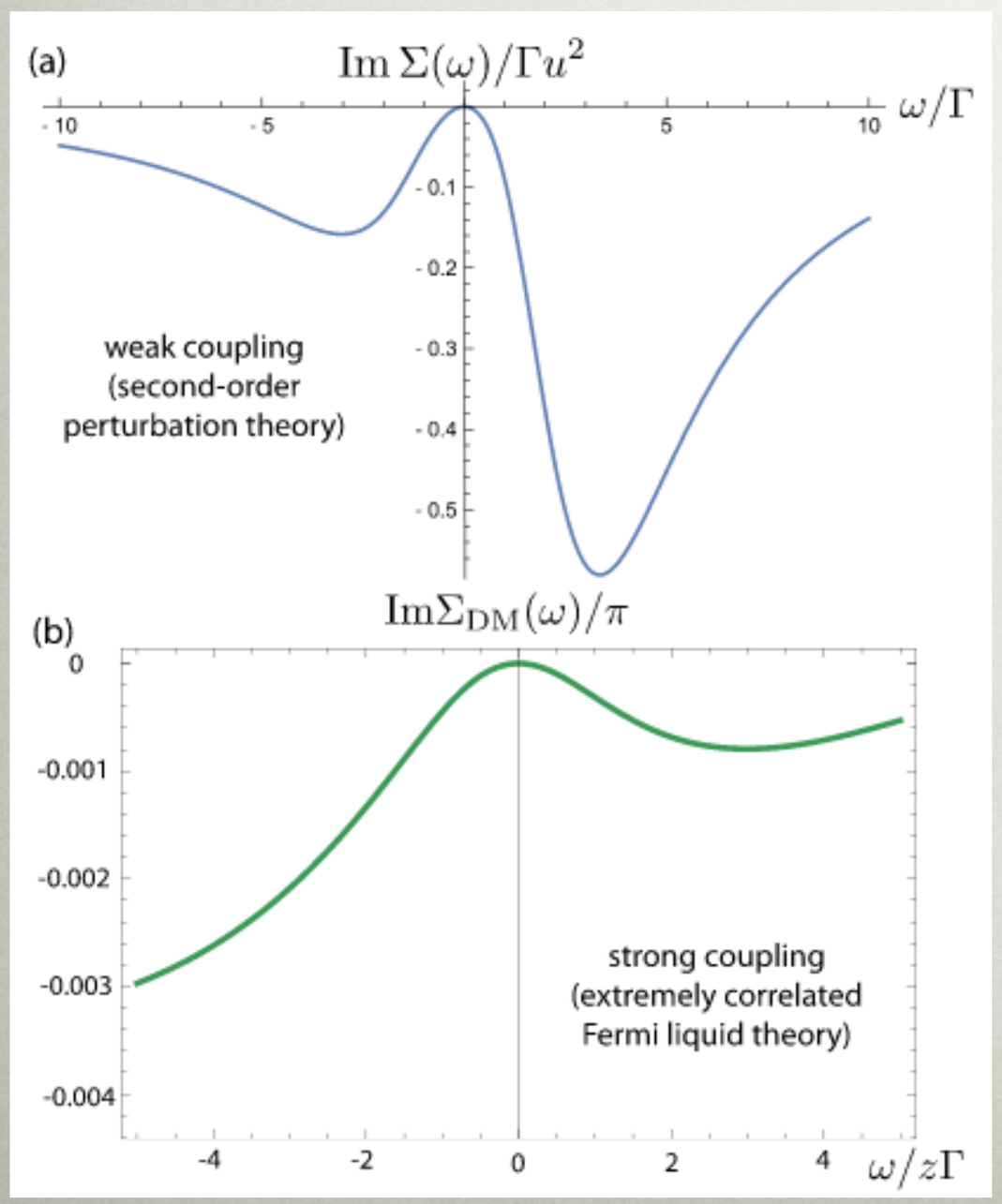
scattering rate for single-particle excitations with energy $|\omega|$ (particles for $\omega>0$, holes for $\omega<0$)

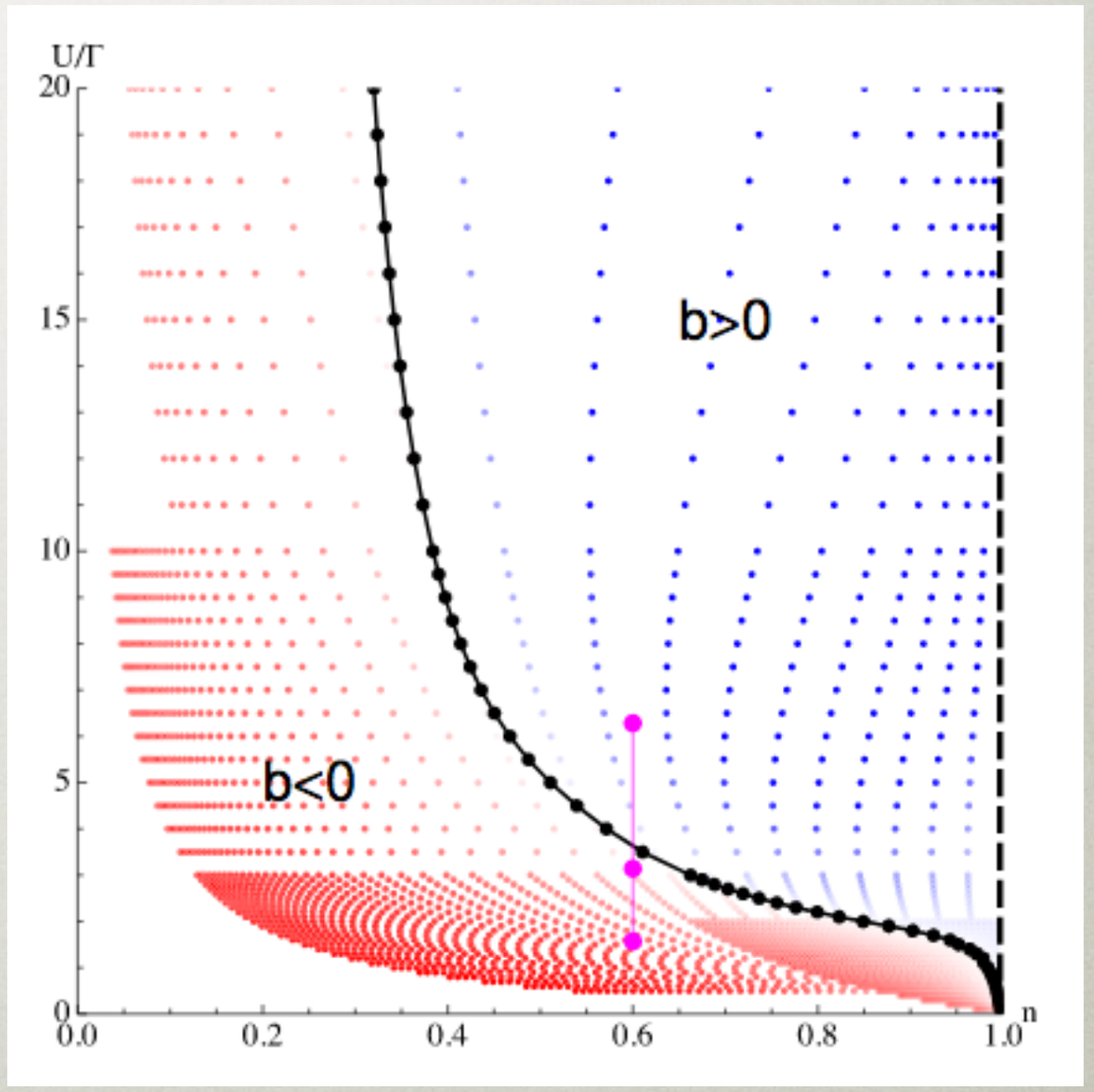
$$1/\tau = Z|\mathrm{Im}\,\Sigma|$$

inverse of the quasiparticle lifetime

Fermi-liquid corrections

$$\mathrm{Im}\,\Sigma(\omega, T) = a(\omega^2 + \pi^2 T^2) + b\omega^3 + c\omega T^2 + \ldots$$





Zitko, Krishnamurthy, Shastry, PRB 2018

Dynamical mean field theory (DMFT)

1) local dynamics: exact calculation!
2) non-local correlations: static mean-field approximation

assumption:

$$\Sigma(k,\omega) \to \Sigma(\omega) \qquad \text{or} \qquad \Sigma_{i,j}(\omega) \to \delta_{ij}\Sigma(\omega)$$

$$G_{\mathbf{k}}(z) = \frac{1}{z + \mu - \epsilon_{\mathbf{k}} - \Sigma(z)}$$

$$G_{\rm loc}(z) = \frac{1}{N}\sum_{\mathbf{k}} \frac{1}{z + \mu - \epsilon_{\mathbf{k}} - \Sigma(z)} = \int \frac{\rho_0(\epsilon)\,{\rm d}\epsilon}{[z + \mu - \Sigma(z)] - \epsilon} = G_0[z + \mu - \Sigma(z)]$$

$$\Delta(z) = z + \mu - \left[G_{\rm loc}^{-1}(z) + \Sigma(z)\right]$$

A. Georges et al., Rev. Mod. Phys. (1996)